CLASSIFICATION OF INDECOMPOSABLE INFINITESIMAL REPRESENTATIONS OF THE LORENTZ GROUP

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Abstract

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Let L be the Lie algebra of the proper Lorentz group, and let L_k be its subalgebra corresponding to the maximal compact subgroup, which is isomorphic to the rotation group of three-dimensional space. Recall ([2]) that a module M is called a **Harish-Chandra module** over the Lie algebra L if, regarded as a module over L_k, it is a direct sum of finite-dimensional submodules: $M = \oplus M_i$. Here each M_i is an irreducible module over L_k, and for every M_{i_0} in the sequence $\{M_i\}$ there exist no more than a finite number equivalent to it.

A module M over the Lie algebra L is defined by specifying six operators $H_+, H_-, H_3, F_+, F_-, F_3$, which satisfy the known commutation relations ([1]). By Δ_1 and Δ_2 we shall denote the Laplace operators. They are expressed in terms of the mappings H and F as follows:

$$\Delta_1 = \frac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3 + F_3;$$
$$\Delta_2 = H_-H_+ - F_-F_+ + H_3^2 - F_3^2 + 2H_3. \tag{1}$$

In the paper the explicit form of the operators H and F acting in the space M of an indecomposable module will be written out.

1. The operators H_+, H_-, H_3 determine in the representation the basis of the subalgebra L_k. It is known that each irreducible representation over L_k is specified by one positive number l, integer or half-integer. In accordance with this, in the decomposition $M = \oplus M_i$ we combine those M_i which correspond to representations with the same l. As a result we obtain a decomposition $M = \oplus R_l$ such that the representation in R_l of the subalgebra L_k is a multiple of an irreducible one. This means that R_l is a finite-dimensional subspace invariant with respect to the operators H_+, H_-, H_3.

By $R_{l,m}$ ($R_{l,m} \subset R_l$) we denote the subspace of eigenvectors of the operator H_3 corresponding to the eigenvalue m:

$$(H_3)_{l,m} = mI_{l,m} \quad (m = -l, -l+1, \dots, l-1, l), \tag{2}$$

where $I_{l,m}$ is the identity operator, and the index l, m by the operator H_3 means that this operator is considered with domain $R_{l,m}$.

Using the explicit form of the representation of the algebra L_k ([1]), one can show that all subspaces $R_{l,m}$ with one and the same l have the same dimension and $R_l = \oplus R_{l,m}$. In this case one can introduce operators E_+ and E_- with the following properties:

$$(E_+)_{l,m} : R_{l,m} \to R_{l,m+1}; \qquad (E_-)_{l,m} : R_{l,m} \to R_{l,m-1}, \tag{3}$$

assuming that $R_{l,l+1} = R_{l,-l-1} = 0$ and

$$(E_-E_+)_{l,m} = I_{l,m} \ (m < l); \qquad (E_+E_-)_{l,m} = I_{l,m} \ (m > -l). \tag{4}$$

The operators H_+ and H_- are expressed through E_+ and E_- by the formulas:

$$(H_+)_{l,m} = \sqrt{(l+m+1)(l-m)}\,(E_+)_{l,m}; \quad (H_-)_{l,m} = \sqrt{(l+m)(l-m+1)}\,(E_-)_{l,m}. \tag{5}$$

Relations (3) and (4) show that the mappings E_+ and E_- define compatible isomorphisms between the spaces $R_{l,m}$ and $R_{l,m+1}$.

2. Operators F_+, F_-, F_3. In order to write expressions for these operators, we introduce the mappings D_+, D_-, Δ

$$(D_+)_{l,m} : R_{l,m} \to R_{l+1,m}; \qquad (D_-)_{l,m} : R_{l,m} \to R_{l-1,m}; \tag{6}$$

$$(\Delta)_{l,m} : R_{l,m} \to R_{l,m}.$$

We shall write out these operators later. Now, with their help, we write expressions for the operators F:

$$(F_3)_{l,m} = \sqrt{l^2 - m^2}(D_-)_{l,m} - md_l(\Delta)_{l,m} - \sqrt{(l+1)^2 - m^2}(D_+)_{l,m}; \tag{7}$$

$$\begin{aligned}(F_+)_{l,m} = &\sqrt{(l-m)(l-m-1)}(D_-)_{l,m+1}(E_+)_{l,m} \\ &- d_l\sqrt{(l-m)(l+m+1)}(E_+)_{l,m}(\Delta)_{l,m} \\ &+ \sqrt{(l+m+1)(l+m+2)}(E_+)_{l+1,m}(D_+)_{l,m};\end{aligned} \tag{8}$$

$$
\begin{aligned}
(F_-)_{l,m} = & -\sqrt{(l+m)(l+m-1)}(D_-)_{l,m-1}(E_-)_{l,m} \\
& - d_l\sqrt{(l+m)(l-m+1)}(E_-)_{l,m}(\Delta)_{l,m} \\
& - \sqrt{(l-m+1)(l-m+2)}(E_-)_{l+1,m}(D_+)_{l,m},
\end{aligned} \tag{9}
$$

where $d_l = -l^{-1}(l+1)^{-1}$.

We shall require of the operators D_+, D_-, Δ that the diagrams be commutative

$$
\begin{array}{ccc}
R_{l-1,m+1} & \xleftarrow{D_-} & R_{l,m+1} \\
\uparrow E_+ & & \uparrow E_+ \\
R_{l-1,m} & \xleftarrow{D_-} & R_{l,m}
\end{array}
\qquad
\begin{array}{ccccc}
R_{l,m+1} & \xleftarrow{\Delta} & R_{l,m+1} & \xrightarrow{D_+} & R_{l+1,m+1} \\
\uparrow E_+ & & \uparrow E_+ & & \uparrow E_+ \\
R_{l,m} & \xleftarrow{\Delta} & R_{l,m} & \xrightarrow{D_+} & R_{l+1,m}
\end{array} \tag{10}
$$

$$
-l+1 \le m < l-1 \qquad\qquad -l \le m < l.
$$

In addition, the same diagrams with the mapping E_- instead of E_+ must also be commutative. Note that, by definition, the mapping $(E_+)_{l,m}$ is an isomorphism (for $m < l$). Consequently, the commutativity of diagrams (10) means that the mappings D_+, D_-, Δ do not depend on the index m. Therefore, in what follows we shall write $(D_+)_l$ instead of $(D_+)_{l,m}$, etc., wherever this cannot cause misunderstanding.

Next we shall require that the operators D_+, D_-, Δ satisfy the relations

$$
(D_-)_l(\Delta)_l = (\Delta)_{l-1}(D_-)_l; \tag{11}
$$

$$
(D_+)_l(\Delta)_l = (\Delta)_{l+1}(D_+)_l; \tag{12}
$$

$$
(2l-1)(D_+)_{l-1}(D_-)_l - (2l+3)(D_-)_{l+1}(D_+)_l = I + l^{-2}(l+1)^{-2}(\Delta)_l^2. \tag{13}
$$

Let us note that of all the relations which the operators D_+, D_-, Δ must satisfy, essentially all except (13) are trivial.

Assertion 1. *Let M be a Harish-Chandra module over the Lie algebra L of the proper Lorentz group. Then the space M is representable in the form of a direct sum of subspaces $R_{l,m}$. Moreover, one can choose mappings $E_+, E_-, D_+, D_-, \Delta$ such that the operators H and F are expressed through them by formulas (5), (7)–(9). Such mappings D, E, Δ must satisfy relations (3)–(4), (10)–(13).*

The converse is also true.

Assertion 2. *If in the space $M = \bigoplus_{l,m} R_{l,m}$ $(m = -l, -l+1, \dots, l)$ operators $D_+, D_-, \Delta, E_+, E_-$ are given which satisfy relations (3), (4), (10)–(13), and through these operators the mappings $H_+, H_-, H_3, F_+, F_-, F_3$ are expressed by formulas (2), (5), (7)–(9), then the operators H and F define a representation of the Lie algebra L of the proper Lorentz group.*

In paper ([2]) it was shown that in an indecomposable module the Laplace operators Δ_1 and Δ_2 have one eigenvalue each, λ_1 and λ_2. Moreover, for each indecomposable module one can specify a number l_0, integer or half-integer $(l_0 \geq 0)$, such that the numbers $\lambda_1, \lambda_2, l_0$ are related by

$$l_0^4 + (1 + \lambda_2) l_0^2 - \lambda_1^2 = 0. \tag{14}$$

The pair (λ_1, λ_2) is called special if one can find a real number l_1 satisfying the identities

$$l_1^2 l_0^2 = -\lambda_1^2; \qquad l_1^2 + l_0^2 = -1 - \lambda_2 \tag{15}$$

and such that the difference $(|l_1| - l_0)$ is a positive integer. A module M with such a pair (λ_1, λ_2) will be called special. Otherwise we shall say that the pair (λ_1, λ_2) and the module M with such a pair are nonspecial.

3. **The operators D_+, D_-, Δ in a nonspecial indecomposable module.** In paper ([2]) it was shown that to every indecomposable nonspecial module there corresponds, and moreover uniquely, a finite-dimensional space P with a linear nilpotent mapping A, where the matrix $[A]$ in some basis has the form of a single Jordan cell.

Theorem 3. *Let M be a nonspecial indecomposable Harish-Chandra module. Then the spaces $R_{l,m}$ $(l_0 \leq l)$ have the same dimension, equal to $\dim P$. At the same time the mappings $(D_-)_{l,m}$ $(l \neq l_0)$ and $(D_+)_{l,m}$ are isomorphisms. In the spaces $R_{l,m}$ one can choose a basis such that the matrices of the mappings D_+, D_-, Δ have the form*

$$[D_+]_{l,m} = [I]; \qquad [\Delta]_{l,m} = [A] + \lambda_1 [I]; \tag{16}$$
$$[D_-]_{l,m} = (l^2 - l_0^2)(4l^2 - 1)^{-1}((1 + \lambda_1 l^{-2} l_0^{-2})[I] + l^{-2} l_0^{-2}(2[A] + [A]^2)),$$

where $[I]$ is the identity matrix, and the matrix $[A]$ has the form of a Jordan cell with zeros on the diagonal.

4. **The operators D_+, D_-, Δ in a special indecomposable module.** In paper ([2]) it was shown that in this case the situation is radically different from that just described. A special indecomposable module, in comparison with an irreducible one, is described by a set of integers, and this set may be arbitrarily long; moreover, in some cases, also by an additional complex number μ. It was shown there that to indecomposable special modules

there corresponds, and moreover bijectively, a pair of finite-dimensional spaces P_1 and P_2 with mappings $d_+: P_1 \to P_2$; $d_-: P_2 \to P_1$; $\delta: P_2 \to P_2$ such that the mappings $d_+ d_-$ and δ are nilpotent and $d_- \delta = \delta d_+ = 0$. Knowing the matrices of the mappings d_+, d_-, δ, one can find the matrices of the mappings D_+, D_-, Δ in the module M. It turns out that in a special indecomposable module M the subspaces $R_{l,m}$ $(l_0 \le l \le l_1 - 1)$ have the same dimension, equal to $\dim P_1$, and the subspaces $R_{l,m}$ $(l_1 \le l)$ also have the same dimension, equal to $\dim P_2$. At the same time, in the subspaces one can choose a basis such that the matrices of the mappings D_+, D_-, Δ have the following form:

$$[D_+]_{l,m} = [I] \quad (l \ne l_1 - 1); \qquad [D_+]_{l_1-1,m} = [d_+]; \tag{17}$$

$$[\Delta]_{l,m} = -l_1 l_0 \big((4l_1^2 - 1)(l_1^2 - l_0^2)^{-1}[d_-][d_+] + [I]\big)^{1/2} \quad (l \le l_1 - 1);$$

$$[\Delta]_{l,m} = -i l_1 l_0 \big((4l_1^2 - 1)(l_1^2 - l_0^2)^{-1}[d_+][d_-] + [\delta] + [I]\big)^{1/2} \quad (l \ge l_1); \tag{18}$$

$$[D_-]_{l,m} = (l^2 - l_0^2) l^{-2} (4l^2 - 1)^{-1} \big(l_1^2 (4l_1^2 - 1)(l_1^2 - l_0^2)^{-1}[d_-][d_+] + (l_1^2 - l^2)[I]\big) \quad (l_0 \le l \le l_1 - 1); \tag{19}$$

$$[D_-]_{l_1,m} = [d_-];$$

$$[D_-]_{l,m} = (l^2 - l_0^2) l^{-2} (4l^2 - 1)^{-1} \big(l_1^2 (4l_1^2 - 1)(l_1^2 - l_0^2)[d_+][d_-] + (l_1^2 - l^2)[I] + l_0^2 (l^2 - l_1^2)(l^2 - l_0^2)[\delta]\big) \quad (l > l_1),$$

where $[I]$ is the identity matrix.

Assertion 4. Let M be an indecomposable special Harish-Chandra module over the Lie algebra L. Then the spaces $R_{l,m}$ $(l_0 \le l \le l_1 - 1)$ have the same dimension, and the spaces $R_{l,m}$ $(l \ge l_1)$ also have the same dimension. Moreover, the mappings $(D_+)_{l,m}$ $(l \ne l_1 - 1)$ and $(D_-)_{l,m}$ $(l \ne l_0;\ l \ne l_1)$ are isomorphisms. The matrices of all mappings D_+, D_-, Δ are expressed in terms of the matrices $[d_+]$, $[d_-]$, $[\delta]$ by formulas (17)–(19). The matrices $[d_+]$, $[d_-]$, $[\delta]$ correspond to operators on the spaces P_1 and P_2:

$$P_1 \xrightarrow{d_+} P_2 \xrightarrow{\delta} P_2 \xrightarrow{d_-} P_1$$

so that the mappings $(d_+ d_-)$ and (δ) are nilpotent, and $\delta d_+ = d_- \delta = 0$.

The canonical form of an indecomposable object which is a pair of spaces with mappings d_+, d_-, δ, is described in paper ([2]). It is of two types. The first type is called an open object, and its invariants are a set of positive integers $[s, n_1, m_1;\ n_2, m_2; \ldots;\ n_k, m_k]$, where $s = 0$ or 1; $n_1 \geq 0$; $n_i > 0$; $m_i > 0$; $m_k \geq -1$. The second type is called a closed object, and its invariants are a set of numbers $[n_1, m_1;\ n_2, m_2; \ldots;\ n_k, m_k;\ \mu;\ N]$, where n_i, m_i, N are positive integers, and μ is an arbitrary complex number. We shall not write out the canonical form of the matrices of the mappings d_+, d_-, δ. We note only that, in the case when these matrices correspond to an open chain, they consist of zeros and ones. If, however, the matrices correspond to a closed chain, then $[d_+]$ and $[d_-]$ have approximately the same form as in the preceding case, while in the matrix $[\delta]$ the number μ occurs in certain positions, and in the remaining positions there are zeros and ones.

Let us describe what the matrices $[d_+]$, $[d_-]$, $[\delta]$ look like in the simplest closed object, to which the set of numbers $[1, 1;\ \mu;\ 1]$ corresponds. In this case the space P_1 has dimension 1, and the space P_2 has dimension 2. In them one can choose a basis l_2 and f_1, f_2 such that

$$d_- f_1 = l_1; \qquad d_- f_2 = 0; \qquad \delta f_1 = \mu f_2; \qquad \delta f_2 = 0; \qquad d_+ l_1 = f_2.$$

Thus the matrices $[d_+]$, $[d_-]$; $[\delta]$; $[d_-][d_+]$; $[d_+][d_-]$ have the form

$$[d_+] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \qquad [d_-] = \begin{pmatrix} 1 & 0 \end{pmatrix}; \qquad [\delta] = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}; \qquad [d_-][d_+] = (0); \qquad [d_+][d_-] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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([1]) I. M. Gel' fand, R. A. Minlos, Z. Ya. Shapiro, *Representations of the Rotation Group and the Lorentz Group*, Moscow, 1958. ([2]) I. M. Gel' fand, V. A. Ponomarev, DAN, **176**, No. 2 (1967).

Note: Figure translations are in progress. See original paper for figures.