ON THE COMPLETE REGULARITY OF STATIONARY PROCESSES

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Abstract

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ON THE COMPLETE REGULARITY OF STATIONARY PROCESSES

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1. Let $x(t)$ be a wide-sense stationary random process with continuous time t and $E\xi(t) = 0$. Denote by H_a^b the closed (in the mean square) linear spans of the random variables $x(t)$, $a \le t \le b$, $-\infty \le a < b \le \infty$.

The process $x(t)$ is called **completely regular** if

$$\sup_{\xi \in H_{-\infty}^0,\ \eta \in H_\tau^\infty} |E\xi\overline{\eta}| = \rho(\tau) \to 0, \qquad \tau \to \infty,$$

where the supremum is taken over those ξ, η for which $E|\xi|^2 = 1$, $E|\eta|^2 = 1$.

In the present note, which adjoins ([1,2]), we continue the study of properties of the spectral density (s.d.) $f(\lambda)$ of the process $x(t)$.

Theorem 1. Let the s.d. $f(\lambda)$ of the process $x(t)$ be representable in the form $f(\lambda) = |B(\lambda)|^2 g(\lambda)$, where:

1) $B(\lambda)$ is a bounded entire function of finite degree ($\le \sigma$);

2) the function $\ln g(\lambda)$ is uniformly continuous, so that

$$\sup_{\lambda,\ |h| \le s} \left| \ln \frac{g(\lambda + h)}{g(\lambda)} \right| = \omega(s) \to 0; \qquad s \to 0$$

3)

$$\int_0^\infty \frac{\omega(s)}{1 + s^2}\, ds < \infty.$$

Then the process $x(t)$ is completely regular, and

$$\rho(\tau) = O\left(\omega\left(\frac{1}{\tau - 2\sigma}\right)\right).$$

Let $B(\lambda)$ be an entire function of finite degree; denote its nonreal zeros by z_j. We shall say that the function $B(z)$ **belongs to the class** $\mathfrak{B}$, if

$$\sup_{-\infty<\lambda<\infty} \sum_j \left| \operatorname{Im} \frac{1}{z_j - \lambda} \right| < \infty.$$

Theorem 2. Let the s.d. $f(\lambda)$ of the process $x(t)$ have the form

$$\frac{|B_1(\lambda)|^2}{|B_2(\lambda)|^2},$$

where B_1, B_2 are entire functions of finite degree, for which

$$\int_{-\infty}^{\infty} \frac{|\ln|B_1(\lambda)||}{1+\lambda^2}\, d\lambda < \infty, \qquad \int_{-\infty}^{\infty} \frac{|\ln|B_2(\lambda)||}{1+\lambda^2}\, d\lambda < \infty,$$

and the function $B_1 \in \mathfrak{B}$. Then

$$\sum_s \rho(2^s) < \infty$$

if and only if $B_2 \in \mathfrak{B}$. Moreover, necessarily

$$\overline{\lim_{\tau}}\, (\rho(\tau))^{1/\tau} \leq e^{-\delta},$$

where

$$\delta = \inf |\operatorname{Im} z_j| > 0,$$

and the infimum is taken over all nonreal zeros z_j of the function $B_2(\lambda)$.

We shall write that $g(\lambda) \in W_2$ if $g(\lambda)/(1+|\lambda|) \in L_2(-\infty, \infty)$. For functions $g(\lambda) \in W_2$, the conjugate function $\tilde{g}(\lambda)$ is defined by the equa-

then

$$\tilde{g}(\lambda) = (\lambda + i) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(u) i \operatorname{sign} u\, e^{i\lambda u}\, du,$$

where $\psi(u)$ is the Fourier transform of the function $g(\lambda)/(\lambda+i)$ (see (3), p. 142).

Theorem 3. Suppose that the s.d. $f(\lambda)$ of the process $x(t)$ can be represented in the form $f(\lambda) = |B(\lambda)|^2 g(\lambda)$, where:

1) $B(\lambda)$ is a square-integrable entire function of finite degree;

2) $g \in W_2,\ g(\lambda) \geq m > 0$;

3) the conjugate function $\tilde{g}(\lambda)$ is continuous everywhere except at the points $\lambda_1, \ldots, \lambda_k$, and is uniformly continuous and bounded on the real line with arbitrary neighborhoods of the points $\lambda_1, \ldots, \lambda_k$ removed;

4)

$$\lim_{\lambda \to \lambda_j} \left| \frac{\tilde{g}(\lambda)}{g(\lambda)} \right| = 0, \qquad j = 1, \ldots, k.$$

Then the process $x(t)$ is completely regular.

The last theorem makes it possible to construct examples of completely regular processes with discontinuous s.d.

2. Let now $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ be a multidimensional completely regular process (see (2)). We shall give here results analogous to the results from (2), but weaker (the terminology from (2) is used without explanation).

The s.d. will now be a nonnegative definite Hermitian matrix-function $\mathbf{f}(\lambda) = \|f_{ij}(\lambda)\|$, which for almost all λ has constant rank $m \leq n$.

Together with $\mathbf{x}(t)$, all one-dimensional processes $x_j(t)$, $j = 1, \ldots, n$, are completely regular. Consequently, for all $f_{jj}(\lambda)$, $j = 1, \ldots, n$, the representation (6) of note (1) holds. Therefore the functions $f_{jj}(\lambda)$ have no discontinuities of the 1st kind, and their vanishing or becoming infinite can occur only as indicated in Corollaries 2 and 3 of (4), i.e., the order of any zero of $f_{jj}(\lambda)$ is an even integer, and for all $\varepsilon > 0$, $\lambda_0 \in (-\infty, \infty)$

$$\lim_{\lambda \to \lambda_0} |\lambda - \lambda_0|^{\varepsilon} f_{jj}(\lambda) = 0.$$

The off-diagonal elements $f_{ij}(\lambda)$ may vanish in an essentially arbitrary way: this follows from Theorem 5 (see below). By virtue of the inequality $|f_{ij}(\lambda)| \leq \sqrt{f_{jj}(\lambda) f_{ii}(\lambda)}$, the elements $f_{ij}(\lambda)$ can become infinite only as slowly as $f_{jj}(\lambda)$. Finally, the following holds.

Theorem 4. If, in some neighborhood of a point λ_0, the functions $f_{ii}(\lambda)$ and $f_{jj}(\lambda)$ are bounded, then the function $f_{ij}(\lambda)$ cannot have a discontinuity of the 1st kind at the point λ_0.

The assumption of boundedness of $f_{ii}(\lambda), f_{jj}(\lambda)$ is essential. Consider, for example, the two-dimensional process $\mathbf{x}(t) = (x_1(t), x_2(t))$ with s.d.

$$\mathbf{f}(\lambda) = \begin{pmatrix} \dfrac{g_{11}(\lambda)}{1+\lambda^2} & \dfrac{\gamma(\lambda)}{1+\lambda^2} \\ \dfrac{\gamma(\lambda)}{1+\lambda^2} & \dfrac{1}{1+\lambda^2} \end{pmatrix},$$

where

$$g_{11}(\lambda) = \ln\left(\frac{\lambda+1}{\lambda}\right)^2 + 1, \quad \lambda > -1/2, \qquad g_{11}(\lambda) = \ln\left(\frac{1-\lambda}{\lambda}\right)^2 + 1, \quad \lambda < -1/2;$$

$\gamma(\lambda) = 0$ for $|\lambda| > 1$, $\gamma(\lambda)$ increases linearly from 0 to 1/2 on $(-1, 0)$ and decreases linearly from 1 to 0 on $(0, 1)$. Relying on Theorem 3, one can prove that the process $\mathbf{x}(t)$ is completely regular, and $\rho(\tau) = O((\ln \tau)^{-1})$.

For processes of full rank the following criterion of complete regularity holds.

Theorem 5. Let the spectral density $f(\lambda)$ of the process $x(t)$ be representable in the form

$$f(\lambda) = B(\lambda)g(\lambda)B^*(\lambda),$$

where $B(\lambda)$ is a square-integrable matrix-valued entire function of finite degree $(\leq \sigma)$, and the matrix function

$$g(\lambda) = \|g_{ij}(\lambda)\|$$

is bounded, uniformly continuous, and its determinant satisfies

$$\det g(\lambda) \geq c > 0.$$

Then the process $x(t)$ is completely regular, and

$$\rho(\tau) = O\left(\max_{i,j} A_{\tau - 2\sigma}(g_{ij})\right).$$

Here $A_s(h)$ denotes the quantity of the best uniform approximation of the function $h(\lambda)$ by bounded entire functions of degree $< s$.

Theorem 6. If

$$\sum \rho(2^s) < \infty,$$

then the spectral density $f(\lambda)$ is uniformly continuous; if

$$\rho(\tau) = O(\tau^{-r-\beta}), \qquad r = 0, 1, \dots, \quad 0 < \beta < 1,$$

then the matrix $f(\lambda)$ has an r-th derivative satisfying a Hölder condition of order β; if

$$\overline{\lim_{\tau}} (\rho(\tau))^{1/\tau} \leq e^{-\delta},$$

then the matrix $f(\lambda)$ is analytically continuable into the strip

$$|\operatorname{Im} z| < \delta, \qquad z = \lambda + i\mu,$$

and is bounded in every strip

$$|\operatorname{Im} z| \leq \delta' < \delta.$$

We shall now show that, just as in the discrete case ([2]), the problem of studying completely regular degenerate processes reduces to an analogous problem for processes of full rank. Let the process $x(t)$ have rank $m < n$. We shall assume that rank m is possessed by the corner matrix

$$g(\lambda) = \|g_{ij}(\lambda)\| = \|f_{ij}(\lambda)\|, \qquad i, j = 1, \dots, m.$$

By $g^{(pq)}(\lambda)$ denote the matrix obtained from $g(\lambda)$ by replacing its p-th row, $p = 1, \dots, m$, by the row

$$(f_{q1}(\lambda), \dots, f_{qm}(\lambda)), \qquad q = m+1, \dots, n.$$

Put

$$a_{pq}(\lambda) = \frac{\det g^{(pq)}(\lambda)}{\det g(\lambda)}.$$

Theorem 7. Let $\rho(\tau; g) \to 0$ as $\tau \to \infty$, and let the functions $a_{pq}(\lambda)$ be representable in the form

$$a_{pq}(\lambda) = \Gamma_{pq}(\lambda)/B_{pq}(\lambda),$$

where Γ_{pq}, B_{pq} are bounded entire functions of finite degree, and moreover all $B_{pq} \in \mathfrak{B}$. Then the process $x(t)$ is completely regular,

$$\rho(\tau; g) \le \rho(\tau; f) \le \rho(\tau/2; g) + O(\exp\{-\tau\delta\}),$$

where

$$\delta = \inf |\operatorname{Im} z_j| > 0,$$

the infimum being taken over all non-real zeros of the functions B_{pq}.

Theorem 8. Let the process $x(t)$ of rank $m < n$ be completely regular. Then the matrix $f(\lambda)$ satisfies the following conditions:

1) $\rho(\tau; g) \to 0, \ \tau \to \infty$;
2) the functions
$$a_{pq}(\lambda) = \Gamma_{pq}(\lambda)/B_{pq}(\lambda),$$
where Γ_{pq}, B_{pq} are bounded entire functions of finite degree.

Let us note that the condition of Theorem 7 that B_{pq} belong to the class $\mathfrak{B}$, although not necessary, is very essential and, generally speaking, cannot be discarded. The corresponding examples are easy to construct using Theorem 2.

3) Let now $x(t)$ be a multidimensional stationary process with discrete time $t = 0, \pm 1, \dots$. Analogues of Theorems 3, 4 hold also in this case. Their formulation, up to obvious changes, coincides with the formulation of Theorems 3, 4, and we shall not give it here.

We formulate one more theorem on the exponentially fast decrease of $\rho(\tau)$ in the case of degenerate processes (see ([2]), item 3).

Theorem 9. In order that

$$\varlimsup_{\tau} (\rho(\tau))^{1/\tau} \leq e^{-\delta}, \qquad \delta > 0,$$

it is necessary and sufficient that the following conditions be fulfilled:

1) the functions $a_{pq}(\lambda)$ be rational functions of $e^{i\lambda}$;
2) the matrix $f(\lambda)$ admit an analytic continuation into the strip

$$|\operatorname{Im} z| < \delta$$

of the complex variable

$$z = \lambda + i\mu.$$

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References

1. I. A. Ibragimov, DAN, 161, No. 1 (1965).
2. I. A. Ibragimov, DAN, 162, No. 5 (1965).
3. N. I. Akhiezer, *Lectures on Approximation Theory*, 1947.
4. I. A. Ibragimov, *Theory of Probability and Its Applications*, 10, 1 (1965).

Note: Figure translations are in progress. See original paper for figures.