ON REGULARIZATION OF THE CAUCHY PROBLEM

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Abstract

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ON REGULARIZATION OF THE CAUCHY PROBLEM

FOR PSEUDODIFFERENTIAL EQUATIONS

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In the present note a modification is carried out of the construction of a regularizer (i.e., the extraction of the nonsmooth part of the solution) for pseudohyperbolic systems (see ([1]))*.

First let us note that we wish to study the nonsmooth part of functions $\varphi(x) \in H_s,\ x \in R^n$. The smoothness of a function in H_s is determined by its belonging to the domain of definition of some power of the operator $\sqrt{-\Delta}$. For convenience we include the function $\varphi(x)$ in a one-parameter family of functions by means of the formula $\psi(x,\tau) = \exp(i\sqrt{-\Delta}\tau)\varphi(x)$. We regard $\psi(x,\tau)$ as a function $\psi(x)$ with values in the Banach space C of functions of the variable τ. The smoothness of such a function is determined by its belonging to the domain of definition of a power of the operator $A = i\partial/\partial\tau$, acting only on one variable. We pass to the construction of a general theory of such functions.

Let H be a complex Hilbert space, and A an unbounded self-adjoint operator with domain of definition $D(A)$ and range $R(A)$, dense in H. Consider the chain of Hilbert spaces $H_s^A,\ s = 1, -1, \dots$, with norms $g\|_s = \|(A+i)^s g\|_H$ (for $s > 0$, $g \in D(A^s)$, and for $s < 0$, H_s^A is the completion of H in the norm $\|\cdot\|_s$).

Let M^n be an n-dimensional Lagrangian manifold in the $2n$-dimensional (Euclidean) phase space p, q ([1,2]), furnished with an infinitely smooth measure $d\sigma$. Consider functions φ on M^n with values in H. On M^n the following "weighted" partition of unity is possible:

$$1 = \sum_{\substack{i \neq i_l \\ j \neq i_l}} \rho_{i_1,\dots,i_k}(y_{i_1,\dots,i_k}); \quad y_{i_1,\dots,i_k} = q_{i_1}, q_{i_2}, \dots, q_{i_k}, p_{i_{k+1}}, \dots, p_{i_n}\ (i_{k-\nu} \neq i_{k+\mu}),$$

$\rho_{i_1,\dots,i_k} \in C^\infty,\ \rho_{i_1,\dots,i_k} = 0$ in a neighborhood of points such that $J_{i_1,\dots,i_k} = dy_{i_1,\dots,i_k}/d\sigma = 0$.

Denote

$$\Delta_{i_1,\dots,i_k} = \partial^2/\partial q_{i_1} + \dots + \partial^2/\partial q_{i_k} + \partial^2/\partial p_{i_{k+1}} + \dots + \partial^2/\partial p_{i_n}.$$

The sequence of norms

$$\left\{ \sum_{\substack{i \neq i_l \\ j \neq i_l}} \left\| \left(\sqrt{-\Delta_{i_1,\dots,i_k}} + i \right)^r \varphi(y_{i_1,\dots,i_k}) \right\|_s^2 J^{-1}_{i_1,\dots,i_k} \rho_{i_1,\dots,i_k} \, dy_{i_1,\dots,i_k} \right.$$

for $r = 1, 2, \dots$ defines the countably normed space v_s, and for $r = 1, 2, \dots,\ s = 1, 2, \dots$ defines the countably normed space v_∞. The completion of the algebraic quotient space v_s/v_∞ in the weak topology

$$g_N - g \in v_N \Rightarrow \lim_{N \to \infty} g_N = g$$

will be denoted by V_s.

We shall now formulate the rule for transforming functions $\varphi \in V_s$ under a canonical change of variables. 1). Under a canonical change which

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* Reported at the International Congress of Mathematicians in Moscow in 1966.

transforms the coordinates $q_{i_1}, q_{i_2}, \dots, q_{i_k}$ into themselves, $\varphi(y_{i_1,\dots,i_k})$ changes as a scalar. 2) In the case of replacing $q_{i_1}, \dots, q_{i_{m-1}} \to p_{i_1}, \dots, p_{i_{m-1}}$, $m - 1 \leqslant k$, we introduce the operator $V^{i_m,\dots,i_k}_{i_1,\dots,i_k} \rho_{i_1,\dots,i_k} \varphi(y_{i_1,\dots,i_k}) = \psi(y_{i_m,\dots,i_k}, \pi/2)$, where $\psi(y_{i_m,\dots,i_k}, t)$ is obtained by the method of successive approximations in powers of $(A + i)^{-1}$ from the equation

$$\begin{aligned} \psi(y_{i_m,\dots,i_k}, t) = \frac{A}{A+i} \rho_{i_1,\dots,i_k}(y_{i_m,\dots,i_k}) \varphi(y_{i_m,\dots,i_k}) + \\ + \frac{1}{A+i} \int_0^t \sqrt{J} \sum_{\mu=1}^{m} \left(\frac{1}{\partial x_{i_\mu}/\partial p_{i_\nu}} \frac{\partial}{\partial p_{i_\nu}} \right)^2 \frac{1}{\sqrt{J}} \psi(y_{i_m,\dots,i_k}, t')\, dt' + \\ + \frac{i}{A+i} \psi(y_{i_m,\dots,i_k}, t), \end{aligned}$$

where $x = q \cos t + p \sin t$; $p = p_{i_1}, \dots, p_{i_m}$; $q = q(y_{i_m,\dots,i_k}) = q_{i_1}, \dots, q_{i_m}$; $J = J_{i_m,\dots,i_k} Dx/Dp$, and all integrals are understood in the sense of bypassing the real poles in the lower half-plane.

Let χ be a canonical atlas whose charts are represented in the form u_j, $y_{i_1,\dots,i_k}$. Let the support of $\varphi_{u_j} \in V_s$ lie in u_j. Denote by

$$U_{u_j}\varphi_{u_j} = \sum_{j_\nu} V^{i_1,\dots,i_k}_{j_1,\dots,j_m}\rho_{j_1,\dots,j_m}\varphi_{u_j}(y_{j_1,\dots,j_m})$$

the transition operator to the coordinates $y_{i_1,\dots,i_k}$.

We now define two zero-dimensional chains. Suppose that on M^n the characteristic class ([1], [2]) is trivial, and $\oint p\,dq = 0$.

Let α^0 be some fixed point on M^n. 1) Put

$$S(u_j,\alpha) \overset{\text{def}}{=} \int_{\alpha^0}^{\alpha} p\,dq - \sum_{\nu=1}^{n-k} p_{i_{k+\nu}} q_{i_{k+\nu}}(\alpha), \qquad \alpha = \alpha(y_{i_1,\dots,i_k}) \in u_j,$$

this zero-dimensional chain with values in the sheaf of functions we shall call the action. 2) Put $\gamma_{u_j} = \operatorname{ind}[\alpha^0,\alpha]$ —the index of inertia $\left\|\partial q_{i_{k+\nu}}/\partial p_{i_{k+\mu}}\right\|_{\nu,\mu=1,\dots,n-k}$, where $\operatorname{ind}[\alpha^0,\alpha]$ is the intersection index of any path from α^0 to α with the singularities of the projection of M^n onto the q-plane (see ([1], [2])). It turns out that γ_u does not depend on α and is an integer zero-dimensional chain.

We now pass to the definition of the canonical operator K^{α^0} (c.o.), mapping the space V_s into the completion W_s in the weak topology ($g_N - g \in K_N \Rightarrow \lim_{N\to\infty} g_N = g$) of the algebraic quotient space $K_s/\bigcap_N K_N$, where K_s is the Hilbert space of functions of q with values in H and with norm

$$\|f\|^2_{K_s} = \int \left\|\left(\sqrt{-\Delta + A^2q^2} + i\right)^s f(q)\right\|^2_H dq:$$

$$K^{\alpha^0}\varphi_{u_j}(\alpha) = \Phi_A^{p_{i_{k+1}},\dots,p_{i_n}} \frac{I^{\gamma_{u_j}}}{\sqrt{|J_{i_1,\dots,i_k}|}} \exp\left[iAS(u_j,\alpha) - \frac{i\pi}{2}\gamma_{u_j}\right] U_{u_j}\varphi_{u_j}(\alpha),$$

where $\Phi_A^{p_{i_{k+1}},\dots,p_{i_n}}$ is the A-Fourier transform in the variables $p_{i_{k+1}},\dots,p_{i_n}$ ([1])*.

* Recall the definition of the A-Fourier transform. Let H^+ (H^-) be a subspace of H, invariant with respect to A, such that A is positive (negative) definite on H^+ (H^-). Obviously, $H = H^+ \oplus H^-$. Denote by I ($I^{1/2}$) a linear operator such that $Ig = g$ ($I^{1/2}g = g$) for $g \in H^+$ and $Ig = -g$ ($I^{1/2}g = ig$) for $g \in H^-$. The operator $|A| = AI = IA$ is positive definite. The operator

$$\Phi_A^{p_1,\dots,p_n}\varphi \overset{\text{def}}{=} (2\pi)^{-n/2}|A|^{n/2}I^{n/2}\int \exp[-ipqA]\varphi(p)\,dp$$

is unitary in K_s.

We have defined a null operator cochain. It turns out that the c.o. does not depend on the canonical atlas. Namely, the following is true:

Theorem 1. *The c.o. is a null cochain with coefficients in the operator bundle.*

Remark. Consider $D^n \subset M^n$. Let $dD^n = M^{n-1}$ be such that the dimensions of the kernels of the projection operators of M^n and M^{n-1} onto the q-plane at points $a \in M^{n-1}$ coincide. Then the c.o. on M^n may be narrowed to D^n.

Example. Let henceforth H be the space of m-component vector-functions of τ with square-integrable modulus, and let $A = i\,\partial/\partial\tau$. It is not hard to verify that, for sufficiently small τ,

$$\exp(i\gamma' - \Delta t)\delta(q-\xi)E = K^\alpha \varphi(a) g(\tau); \quad g(\tau) = \frac{(n-1)!(\tau - i0)^{-n/2}}{\frac{n}{2}!(2\pi)^{n/2}},$$

where E is the identity $m \times m$ matrix; M^n is a cylinder; $|p| = 1$; $q = \xi + pt'$ (ξ is fixed, t' is a parameter); $\varphi(a) = \varphi(t')$ is finite and equal to unity in a neighborhood of $t' = 0$; $d\sigma = dt'\,d\omega$, where $d\omega$ is angular measure on the sphere $|p| = 1$. Denote by M^n_ξ the support of $\varphi(a)$ on this manifold.

We pass to the pseudohyperbolic equation.

Let $L(p,q,t)$ be a $(2n+1)$-parameter $m \times m$ matrix possessing the following properties: 1) the coefficients of the matrices $L(p,q,t)$;

$$L^0(p,q,t) \overset{\text{def}}{=} \lim_{h\to 0} hL(h^{-1}p,q,t) \quad \text{and} \quad L^1(p,q,t) \overset{\text{def}}{=} \lim_{h\to 0} h\big(hL(h^{-1}p,q,t) - \\ - L^0(p,q,t)\big)$$

belong to C^∞. 2) The eigenvalues $\lambda^1(p,q,t), \dots, \lambda^m(p,q,t)$ of the matrix L^0 are real, their multiplicity is constant, and there are no conical subspaces. 3) The solution of the system $\dot p = -\lambda^i_q,\ \dot q = \lambda^i_p,\ i = 1, \dots, m$, exists globally.*

Consider the problem

$$i\,\partial u/\partial t + \hat L u = f(q,t), \qquad f(q,t) \in H_s; \tag{1}$$

$$u\big|_{t=0} = 0, \tag{2}$$

where $\hat L$ is a pseudodifferential operator with symbol $L(p,q,t)$. Notation: M^{n+1}_i is the Lagrangian manifold with boundary formed by trajectories of the system $\dot p = -\lambda^i_q,\ \dot q = \lambda^i_p,\ 0 \le t \le t_1$, with initial conditions on M^n_ξ: $M^{n+1}_i\big|_{t=0} = M^n_\xi$; K^α_i is the c.o. corresponding to the manifold M^{n+1}_i, the measure $d\sigma = dt\,dt'\,d\omega$, and some fixed partition of unity; L^i is the matrix adjoint to the matrix composed of the complementary minors of $L^0 - \lambda^i$, multiplied by

$$\prod_{j\ne i}(\lambda^i - \lambda^j)^{-1},$$

$$L_0^i \stackrel{\text{def}}{\equiv} L^i\big|_{t=0};$$

R^i **is the matrix inverse to** $L^0 - \lambda^i$ **on the range** $R(L^0 - \lambda^i)$ **of its values;**

$$P^i = \left[\sum_{j=0} L^i_{p_j} L^0_{q_j} + L^1 L^i\right]_{p,q\in M_i^{n+1},\, p_0=\lambda^i_{q_0=t}}$$

is the polarization matrix*; $T^j = id/dt + P^j$ is an operator on M_j^{n+1}, defined on functions equal to zero at $t = 0$ (on M_ξ^n); L_1, L_2 are certain infinitely differentiable operators in $W_{-[n/2]-1}$, whose analytic expression we do not

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* It is sufficient that the derivative of λ^i with respect to p in the direction q grow no faster than

$$|q| \ln |q| \ln\ln |q| \cdots \underbrace{\ln\ln\cdots\ln}_{k} |q|$$

for some k.

** We note that

$$\hat{L}\psi(q) = \Phi_A^{*\,q_1\cdots q_n} L(Ap, q, t)\Phi_A^{p_1\cdots p_n}\psi(q).$$

*** This matrix has a substantial physical meaning. It shows the deviations of the spin and dissipative character in the j-th component in the regulator R from the regulator of the classical self-adjoint equation. As examples, $i\,\partial\psi/\partial t = \lambda^j(\hat{p}, q, t)\psi,\ \hat{p} = i\partial/\partial q$, where $\lambda^j(\hat{p}, q, t)$ is a pseudodifferential operator in Weyl' s sense with symbol $\lambda^j(p, q, t)$.

we shall not present it here. The sign $\doteq$ denotes equality in $W_{-[n/2]-1}$, i.e., up to a function from $\bigcap_N K_N$.

Theorem 2. *Under conditions 1)–3), the solution of problem (1), (2) exists and is unique in the space $\bar{H}_s$ and, up to arbitrarily smooth functions, is representable in the form $u = Rf(q, t)$, where the operator R is written out below.*

Let us outline the proof.

Making the substitution $u \doteq K_i^{\alpha^0} G^i(a, t)g(\tau)$, we obtain

$$\left(i\frac{\partial}{\partial t} + \hat{L}\right) K_i^{\alpha^0} G^i(a, t)g(\tau) \doteq K_i^{\alpha^0}\left[(L^0 - \lambda^i) + \frac{1}{A+i}\hat{L}_1 + \frac{1}{(A+i)^2}\hat{L}_2\right]\times$$

$$\times\, G^i(a, t)g(\tau).$$

Expanding $G^i(a, t)$ in powers of $(A + i)^{-1}$ under the condition $G^i|_{t=0} = L_0^i\varphi(t')$, we obtain [1]

$$G^j \doteq \sum_{k=0}^{\infty} (A+i)^{-k} f_k^j,$$

where

$$f_k^j = \left(1 - B_j^{-1} L^j \hat{L}_1\right) R^j (\hat{L}_1 f_{k-1}^j + \hat{L}_2 f_{k-2}^j) - B_j^{-1} L^j \hat{L}_2 f_{k-1}^j; \quad B_j = T_j + \frac{1}{2} \sum_k \lambda_{p_k q_k};$$

$$f_0^j = \exp\left[\frac{1}{2} \int_0^t \sum_k \lambda_{p_k q_k}^j \, dt\right] T_j^{-1} P^j L_0^j \varphi(t').$$

The sum $\sum_{i=1}^m K_i^{\alpha^0} G^i(a,t) g(\tau)$ gives, in the form of a series in powers of $(A+i)^{-1}$, the solution of equation (1) for $f(q,t)=0$ and under the condition $u|_{t=0} = \exp[i\sqrt{-\Delta}\tau]\delta(q-\xi)E$, as an operator from $W_{-[n/2]-1}$ into $W_{-[n/2]-1}$.

Take a sufficiently large partial sum of the resulting series and denote the resulting $m \times m$ matrix by $G(q,\xi,t,\tau)$. The operator

$$RF(q,t) = \int_0^t dt^* \int G(q,\xi,t-t^*,\tau) \exp[-i\sqrt{-\Delta}\tau] F(\xi,t^*) \, d\xi$$

is a regularizer of the solution of equation (1): obviously,

$$\left(i\partial/\partial t + \hat{L}\right) RF(q,t) = F + \nu F,$$

where ν is a Volterra operator with an arbitrarily smooth kernel.

Hence for any $f(q,t) \in H_s$ there exists a solution of problem (1)–(2) (namely, $u = R(1+\nu)^{-1} f(q,t)$), which, up to sufficiently smooth functions, is representable in the form $Rf(q,t)$. The analogous assertion for the adjoint operator leads to uniqueness of the solution of problem (1), (2). Since an integral representation of the solution has been found here, its smoothness is investigated analogously to how this is done for equations with constant coefficients. Analogous existence and uniqueness theorems are obtained in a substantially more general situation.

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References Cited

[1] V. P. Maslov, *Perturbation Theory and Asymptotic Methods*, 1965. [2] V. I. Arnold, *Functional Analysis*, 1, no. 1, 1967, p. 1.

Note: Figure translations are in progress. See original paper for figures.