CALCULATION OF FLOW IN A LAVAL NOZZLE

HYDROMECHANICS

1967

SovietRxiv

Abstract

Full Text

UDC 533.697.4

HYDROMECHANICS

U. G. PIRUMOV

CALCULATION OF FLOW IN A LAVAL NOZZLE

(Presented by Academician G. I. Petrov, 25 X 1966)

In the present paper the inverse problem of the theory of the Laval nozzle is considered; it is a Cauchy problem for the equations of gas dynamics and consists in determining the streamlines and the flow parameters from the known distribution of velocity on the axis of symmetry. In the solutions currently available ([1−6]), mainly a small neighborhood of the center of the nozzle has been investigated analytically. Such solutions do not make it possible to study the flow in the subsonic and transonic parts of nozzles, in which the flow differs noticeably from one-dimensional flow. Below a numerical method of solution is described and a stable difference scheme is proposed, making it possible to calculate the subsonic, transonic, and supersonic regions of flow in Laval nozzles with large gradients of the gas-dynamic parameters.

§ 1. On the axis of symmetry of the nozzle the velocity distribution is prescribed in the subsonic, transonic, and supersonic regions. According to the Cauchy-Kovalevskaya theorem, for analytic initial data for the equations of gas dynamics there exists, in some neighborhood of the axis, a unique analytic solution of the Cauchy problem. In the numerical solution of the Cauchy problem difficulties arise. In the elliptic region, in the general case the Cauchy problem is ill-posed, although, if a class of analytic functions is considered, then in a bounded domain the problem becomes well-posed ([7]). Nevertheless, even for analytic initial data in the subsonic region of the nozzle, where the equations of gas dynamics are elliptic, with an unsuccessfully chosen difference scheme the ill-posedness of the Cauchy problem manifests itself in an extremely rapid growth of round-off errors, which inevitably arise in a numerical solution. On the other hand, in the general case in a hyperbolic region the ratio of the steps in the difference scheme must be such that the domain of influence of the approximating system does not go beyond the domain of influence of the original system of differential equations ([8]); however, in the class of analytic functions the ratio of the steps in the difference scheme may be arbitrary ([9]).

Let us consider a system of gas-dynamic equations describing irrotational isentropic flow of an ideal gas with a constant adiabatic exponent. As independent variables we use the stream function ψ and the Cartesian coordinate x. In these variables the system of equations of gas dynamics has the form ([10])

$$\frac{\partial y^2/2^j}{\partial \psi} = \frac{1}{\rho u}, \qquad \frac{\partial y}{\partial x} = \frac{v}{u}, \qquad \frac{\partial p}{\partial \psi} = -\frac{k}{y^j}\frac{\partial v}{\partial x},$$
$$\rho = p^{1/k}, \qquad u = \left(\frac{k+1}{k-1} - \frac{2p^{(k-1)/k}}{k-1} - v^2\right)^{1/2}. \tag{1}$$

Here u and v are the projections of the velocity vector on the axes x and y of the Cartesian coordinate system, referred to a_*, the critical speed of sound; p and ρ are the pressure and density, referred to the pressure and density at $w = a_*$; k is the ratio of specific heats; and $j = 0$ or 1 for the plane or axisymmetric case, respectively.

Let us write down the difference scheme used in the present work, corresponding to system (1). Suppose that on the n-th layer $\psi_n =$ const all flow parameters are known at i points, which in the general case are not equidistant from one another. Then the parameters on the $(n+1)$-st layer $\psi_{n+1} =$ const are determined by the formulas

$$\left[y^2_{i(n+1)}\right]^{(\nu)} = y^2_{in} + 2^{j-1}\Delta\psi\left[\frac{1}{(\rho u)_{in}} + \frac{1}{(\rho u)^{(\nu-1)}_{i(n+1)}}\right]; \tag{2}$$

$$p^{(\nu)}_{i(n+1)} = p_{in} - \frac{k\Delta\psi}{2}\left[\frac{1}{y^j_{in}}\left(\frac{\partial v}{\partial x}\right)_{in} + \frac{1}{\left(y^j_{i(n+1)}\right)^{(\nu-1)}}\left(\frac{\partial v}{\partial x}\right)^{(\nu-1)}_{i(n+1)}\right]; \tag{3}$$

$$v^{(\nu)}_{i(n+1)} = \frac{(\partial y/\partial x)^{(\nu)}_{i(n+1)}}{\sqrt{1 + \left[(\partial y/\partial x)^2_{i(n+1)}\right]^{(\nu)}}}\left[\frac{k+1}{k-1} - \frac{2\left(p^{(k-1)/k}_{i(n+1)}\right)^{(\nu)}}{k-1}\right]^{1/2} \tag{4}$$

$$\rho^{(\nu)}_{i(n+1)} = \left[p^{1/k}_{i(n+1)}\right]^{(\nu)}, \qquad u^{(\nu)}_{i(n+1)} = \left[\frac{k+1}{k-1} - \frac{2\left(p^{(k-1)/k}_{i(n+1)}\right)^{(\nu)}}{k-1} - \left(v^2_{i(n+1)}\right)^{(\nu)}\right]^{1/2}. \tag{5}$$

Here ν is the iteration number, and $\Delta\psi$ is the integration step along the normal to the streamlines. The approximation error in this direction is of order $(\Delta\psi)^2$.

Fig. 1. Dependence of the vertical component of velocity on the length of the streamline

Fig. 1. Dependence of the vertical component of velocity on the length of the streamline

Figure 1: Fig. 1. Dependence of the vertical component of velocity on the length of the streamline

The iterative process for computing the parameters by formulas (2)–(5) is as follows. In the first approximation, by formulas (2) and (3), at all points of the n-th layer the quantities $y_{i(n+1)}^{(1)}$ and $p_{i(n+1)}^{(1)}$ are calculated; here the quantities with index $(\nu-1)$ are taken equal to the corresponding quantities on the n-th layer. Then, from the found values of y, $(\partial y/\partial x)_{i(n+1)}^{(1)}$ is determined, and by formula (4) $v_{i(n+1)}^{(1)}$ is determined; finally, by formulas (5) the values $\rho_{i(n+1)}^{(1)}$ and $u_{i(n+1)}^{(1)}$ are determined. The parameters in subsequent iterations are calculated in an analogous manner.

Let us turn to the difference representation of the derivatives $\partial v/\partial x$ and $\partial y/\partial x$. Figure 1 presents a typical dependence of v on the length of the nozzle. Since the gradients of the function v differ greatly in different regions, it is natural to choose the steps of the difference grid as variable on the layer. Thus, in regions I and III the steps should be large, since, because of the small gradients in this region, the approximation errors will be small, as will the round-off errors due to the large steps of the difference grid. In regions II and IV the step of the difference grid should be sufficiently small. To compute the derivatives $\partial v/\partial x$ and $\partial y/\partial x$, the present work used a three-point scheme which, as special calculations showed, is more stable than multipoint schemes.

§ 2. Using the three-point scheme described above with a variable step on the layer, for the axisymmetric case, as a result of the numerical solution of the inverse problem, flows with a rectilinear and a curvilinear transition surface were calculated. Calculations were carried out for the following velocity distributions along the axis:

$$w = w_\infty + \frac{1-w_\infty}{1+Ax^2}, \qquad w_\infty = 0,1, \qquad A = 10; \tag{6}$$

$$w = 1 + \frac{2(1-w_\infty)}{\pi} \operatorname{arc\,tg}[e^{-ax} - e^{bx}], \qquad w_\infty = 0,1, \qquad a = 2,3, \qquad b = 0,2; \tag{7}$$

$$w = 1 + \frac{e^{-x/b} - 1}{e^{-x/b}/(\bar{w}_\infty - 1) + 1/(1-w_\infty)}, \qquad w_\infty = 0,1, \qquad \bar{w}_\infty = 1,9, \qquad \frac{1}{b} = 3,5. \tag{8}$$

Figure 2

Figure 2: Figure 2

Nozzle profile and pressure plot

Figure 3: Nozzle profile and pressure plot

Fig. 2. Velocity distribution (6). $k = 1.4$. a–g–streamlines: $\psi = 0.08$ (a), 0.06 (b), 0.04 (c), 0.02 (d). 1–10–lines $w = \text{const}$: $w = 0.11$ (1), 0.12 (2), 0.15 (3), 0.2 (4), 0.3 (5), 0.5 (6), 0.8 (7), 1.0 (8), 1.05 (9), 1.10 (10). I, II–velocity distribution along the streamline $\psi = 0.06$ as computed by the author' s method (I) and by one-dimensional theory (II); III–velocity distribution along the axis.

Fig. 3. Velocity distribution (8). $k = 1.4$. a–d–streamlines: $\psi = 0.08$, $1 - u = 1.1\%$ (a), 0.06, 0.05% (b), 0.04, 0.19% (c), 0.02, 0.04% (d), 0.01, 0.0064% (e). 1–16–lines $w = \text{const}$: $w = 0.105$ (1), 0.12 (2), 0.15 (3), 0.2 (4), 0.3 (5), 0.5 (6), 0.7 (7), 0.9 (8), 1.2 (9), 1.4 (10), 1.5 (11), 1.6 (12), 1.7 (13), 1.8 (14), 1.85 (15), 1.87 (16). I, II, III–same as in Fig. 2.

For the velocity distribution (6) the sonic line is rectilinear, since in order for the sonic line to be rectilinear it is necessary and sufficient that, at $w = 1$, $\partial w/\partial x = 0$ (11). In Fig. 2, for this case, streamlines and lines $w = \text{const}$ are presented, as well as the velocity distribution on the streamline with $\psi = 0.06$. As can be seen from Fig. 2, the flow differs noticeably from one-dimensional flow, especially at large values of the velocity. In the neighborhood of the rectilinear sonic line, located in the plane $x = 0$, for $y > 0.25$ there arises a local supersonic zone and a second sonic line (12). Between the second sonic line and the rectilinear sonic line the gas is initially accelerated and then decelerated, which leads to the appearance in this region of a positive pressure gradient. It should be noted that, when the shape of the contour changes abruptly in the transonic region, a positive pressure gradient may also arise in nozzles with a curvilinear transition surface. Such a result is obtained by calculating the flow field for the velocity distribution (7).

The results of the calculation for the velocity distribution (8) are presented in Fig. 3, which shows a family of streamlines, lines $w = \text{const}$, the sonic line, and the line $\theta = 0$ (θ is the angle of inclination of the velocity to the axis). In addition, in the same figure, for each streamline the value of $(1 - \mu)\%$ is indicated, where μ is the discharge coefficient. The same figure gives the velocity distribution on the streamline with $\psi = 0.06$. The sonic line and the line $\theta = 0$ deviate upstream from the center of the nozzle in such a way that the sonic point is located beyond the minimum section of the nozzle. Characteristic of

Figure 3

Figure 4: Figure 3

this case is the fact that, for $x \geq 1.5$, the streamlines are practically rectilinear and parallel to the axis. It follows from the calculation results that on all streamlines, beginning with $\psi = 0.03$, a positive pressure gradient arises at the beginning of the cylindrical section of the nozzle contour. The coordinates of the streamline with $\psi = 0.06$ are very close to the coordinates of the subsonic part of a nozzle whose neighborhood of the critical section is made in the form of an arc of a circle with radius $R_2 = 1.6r_*$, to which a conical section with angle $\theta_{\text{in}} = 35°$ is faired, joined by a radius $R_1 = r_{\text{in}}$ to the cylindrical part (see Fig. 3).

The numerical solution for the velocity distribution along the axis (6) was compared with several asymptotic solutions, in particular with the solution obtained by expanding the unknown parameters in a series in the neighborhood of the axis of symmetry, in the neighborhood of the rectilinear sonic line, and with the solution of the inverse problem of Laval-nozzle theory for an incompressible fluid. In the region of convergence of the asymptotic solutions, good agreement between the numerical solution and the asymptotic solutions was obtained.

The method proposed in the present work can be used for calculating multilayer flows with different physical properties and for calculating equilibrium and nonequilibrium flows.

Received
19 X 1966

REFERENCES CITED

1. Th. Meyer, *Forschungshefte*, **62** (1908).
2. G. I. Taylor, Great Britain Aeronautical Res. Com. Rep. and Memoranda, No. 1381 (1930).
3. F. I. Frankl, *Izv. AN SSSR, ser. matem.*, **9**, No. 5 (1945).
4. S. V. Falkovich, *PMM*, **10**, issue 4 (1946).
5. Th. von Kármán, *J. Math. and Phys.*, **26**, 182 (1947); Russian transl. in the collection *Gas Dynamics*, IL, 1950, p. 41.
6. K. G. Guderley, *Theory of Transonic Flows*, IL, 1960.
7. M. M. Lavrent' ev, *Izv. AN SSSR, ser. matem.*, **20**, 6 (1956).
8. I. S. Berezin, N. P. Zhidkov, *Methods of Computation*, M., 1959.
9. G. Dahlquist, *Math. Scand.*, **2** (1954).

10. N. E. Kochin, I. A. Kibel' , N. V. Roze, *Theoretical Hydromechanics*, Part II, M., 1963.

11. H. Görtler, *Math. u. Mech.*, **19**, 325 (1939).

12. L. V. Ovsyannikov, *Transactions of the Leningrad Air Force Engineering Academy*, **1**, issue 33 (1950).

Note: Figure translations are in progress. See original paper for figures.