DUALITY OF FUNCTORS

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Abstract

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MATHEMATICS

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DUALITY OF FUNCTORS

IN THE CATEGORY OF LOCALLY CONVEX SPACES

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In this note we consider functors in the category L of separated locally convex spaces (the morphisms are taken to be linear mappings continuous on bicompact sets). Define the functors $H(X, Y)$ and $X \otimes Y$ by assigning to each pair of spaces X, Y the space $H(X, Y)$ of all morphisms from X to Y, endowed with the topology of uniform convergence on bicompact sets, and the space $X \otimes Y$ —the vector tensor product of these spaces, endowed with the strongest of the locally convex topologies that coincide with the projective topology (^{6}) of the tensor product on sets of the form $\varphi(K_1 \times K_2)$, where $\varphi : X \times Y \to X \otimes Y$, and K_1 and K_2 are bicompact subsets of X and Y, respectively. In the category under consideration, the space R of real numbers is simultaneously an integral and a cointegral object (^{5}).

The functor dual to a functor F is called the functor DF, which to each space A assigns the set $\operatorname{Hom}(F, \Sigma_A)$ of all mappings of the functor F into the functor Σ_A, endowed with the weakest of the topologies for which the mapping

$$\operatorname{Hom}(F, \Sigma_A) \to H(F(X), \Sigma_A(X))$$

is continuous for every X (the functor Σ_A is defined by the relation $\Sigma_A(X) = A \otimes X$). In (^{2}) it is shown that the category considered here, with the functors $H(X, Y)$ and $X \otimes Y$ defined above, is a D-category (^{1}); the notion of a dual functor given here coincides with that defined in (^{1}).

In the present note we give a certain description of the dual functor in the category L. Next, we study a class of functors defined by Banach spaces of numerical sequences. For a functor n from this class, the functors Dn and D^2n are computed, and the reflexivity of the functor Dn is shown.

We first study the operation of passing to the conjugate space in our category and the connection of this operation with the duality of functors. (By the space conjugate to a space X we mean, in the category L, the space

$$\overline{X} = H(X, R).$$

) We shall call the extremal topology of the space $\overline{X}$ the strongest of the topologies in it that are isomorphic, in the sense of our category, to the original topology of the space X. The following assertions hold.

1. *The bicompact sets in the space $\overline{X}$ are precisely the weakly closed equicontinuous sets in the extremal topology of the space X ([7]), and only these.*
2. *The completion of the space X with respect to its extremal topology is the space*

$$\overline{\overline{X}} \ ({}^8),$$

endowed with its extremal topology.

The latter means, in particular, that the morphism

$$X \to \overline{\overline{X}}$$

is an isomorphic embedding in the sense of the category L (we denote this by

$$X \subseteq \overline{\overline{X}}).$$

Defining naturally the morphism

$$\lambda : DF(A) \to F(\overline{A})$$

(see ([1])), one can prove, analogously to Theorem $1'$ of ([1]), that *for any functor F and any space A, the morphism*

$$\lambda : DF(A) \to F(\overline{A})$$

is an isomorphic embedding.

This proposition describes the topology of the space $DF(A)$. We shall now characterize the set of points of the space $F(\overline{A})$ that correspond to elements of the space $DF(A)$.

Let Λ be an arbitrary directed set, and let $C_0^\Lambda(\overline{A})$ be the space of sequences $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of morphisms $\varphi_\lambda \in \overline{A}$ weakly converging to zero, endowed with the topology of coordinatewise convergence in the finest topology of the space $\overline{A}$. The projection morphism $p_\lambda : C_0^\Lambda(\overline{A}) \to \overline{A}$ onto the coordinate λ defines the morphism $F(p_\lambda) : F(C_0^\Lambda(\overline{A})) \to F(\overline{A})$. Fix a point $\xi \in F(C_0^\Lambda(A))$, and call the sequence of its images under these mappings $\{F(p_\lambda)(\xi)\}$ a special sequence in $F(\overline{A})$. We shall call the special topology in $F(\overline{A})$ the strongest of the locally convex topologies in which the special sequences corresponding to all possible Λ and ξ converge to zero.

Proposition 1. *For any functor F and any space A, the space $DF(A)$ consists of the morphisms $F(\overline{A}) \to R$ that are continuous in the special topology.*

This assertion may be derived from the description of the dual functor obtained in (9).

Let n be the complete (4) minimal normed ideal in the space m of bounded numerical sequences (3). Define the functor $n(X)$ as follows. Let the space $n(X)$ consist of sequences $\{x_i\}$ in X such that, for every seminorm $p \in \Gamma$ (Γ is the family of seminorms determining the finest topology in X), the sequence $\{p(x_i)\} \in n$. For each seminorm $p \in \Gamma$, define a seminorm $\overline{p}$ in $n(X)$: $\overline{p}(\{x_i\}) = \|\{p(x_i)\}\|$. The topology in $n(X)$ is defined by the family $\overline{\Gamma}$ of all seminorms $\overline{p}$ obtained in this way. It can be proved that the functor n is admissible (1).

With respect to the functor Dn, dual to the functor n, the following is true.

Proposition 2. *For every space A, the space $Dn(A)$ is the space of finitary sequences in A.*

We give the main points of the proof. Let α be a mapping of the functor n into the functor Σ_A. Consider the morphism $\alpha_R : n(R) \to A$ which it realizes for $X = R$, and denote the images of the points $e_i \in n(R) = n$ by $a_i = \alpha_R(e_i)$. It is easy to see that the mapping of functors α sends each element $\{x_i\} \in n(X)$ to

$$\sum_{i=1}^{\infty} a_i \otimes x_i \in A \otimes X,$$

so that it is completely determined by the sequence $\{a_i\}$. Using Proposition 1, one can show that, in the case $A = R$, the sequence $\{a_i\}$ cannot have infinitely many nonzero elements. On this basis one can conclude that, for any space A, the sequence $\{a_i\}$ is either finitary or does not lie in any finite-dimensional subspace. If the latter occurs, select a subsequence $\{a_{i_k}\}$ of linearly independent nonzero elements. It is not difficult to verify that it, like every subsequence of the sequence $\{a_i\}$, gives rise to a mapping of functors $n \to \Sigma_A$. Construct a morphism $\varphi : A \to \Pi$ (Π is the space of numerical sequences with coordinatewise convergence) such that $\varphi(a_{i_k}) = e_k$. Then the morphism $Dn(\varphi) : Dn(A) \to Dn(\Pi)$ assigns to the sequence $\{a_{i_k}\} \in Dn(A)$ the sequence $\{e_k\}$, which, by virtue of this, would have to belong to the space $Dn(\Pi)$. But the latter is impossible, since the sequence $\{e_k\}$ does not give rise to a mapping of functors $n \to \Sigma_\Pi$ (for example, for $\{e_k\} \in n(\Pi)$ the series

$$\sum_{i=1}^{\infty} e_k \otimes e_k$$

does not converge in $\Pi \otimes \Pi$). Consequently, for any space A, the sequence $\{a_i\} \in Dn(A)$ is finitary. It is easy to see that the converse is also true.

Proposition 3. *The functor D^2n is algebraically isomorphic to the functor n.*

Let us outline the proof of this proposition. One can verify that the natural mapping of functors $\chi : F \to D^2F$ (1) is one-to-one.

significant for the functor $\mathfrak{n}$. Therefore it suffices for us to show that for every A the morphism $\kappa_A : \mathfrak{n}(A) \to D^2\mathfrak{n}(A)$ is an epimorphism. Let $a \in D^2\mathfrak{n}(A)$; then a is a mapping of functors $D\mathfrak{n} \to \Sigma_A$. As in the preceding case, it is completely determined by a sequence of points $a_i = a_R(e_i)$ such that the element $\{x_i\} \in D\mathfrak{n}(X)$ is mapped to $\Sigma a_i \otimes x_i \in A \otimes X$. It is necessary to show that $\{a_i\} \in \mathfrak{n}(A)$, i.e. that for every seminorm p, continuous in the extremal topology of the space A, the sequence $\{p(a_i)\}$ is summable with every sequence $\{\lambda_i\} \in \mathfrak{n}'$ ($\mathfrak{n}'$ is the space of sequences summable with every sequence $\{\nu_i\} \in \mathfrak{n}$).

Construct a sequence $\{f_i\}$ of morphisms $f_i \in \bar{A}$ such that $f_i(a_i) = p(a_i)$ and $|f_i(x)| \leq p(x)$ for all $x \in A$. One can show that for any $\{\lambda_i\} \in \mathfrak{n}'$ the sequence $\Phi = \{\lambda_i f_i\} \in H(\mathfrak{n}(\bar{A}), R)$, and that there it is the limit of its finite subsequences Φ_n. The latter belong to the space $D\mathfrak{n}(\bar{A})$ and, since $D\mathfrak{n}(\bar{A}) \subseteq H(\mathfrak{n}(\bar{A}), R)$, form there a sequence fundamental in the extremal topology. Consider the morphism $\gamma : D\mathfrak{n}(\bar{A}) \to R$ (the composition of the morphisms α_A:

$$D\mathfrak{n}(\bar{A}) \to A \otimes \bar{A} \quad \text{and} \quad A \otimes \bar{A} \to R),$$

for it

$$\gamma(\Phi_n) = \sum_{i=1}^{n} \lambda_i p(a_i).$$

The sums

$$\sum_{i=1}^{n} \lambda_i p(a_i),$$

because the sequence $\{\Phi_n\}$ is fundamental in the extremal topology of the space $D\mathfrak{n}(\bar{A})$, and γ is a mapping continuous in this topology, form a fundamental sequence in the space R, so that

$$\sum_{i=1}^{\infty} \lambda_i p(a_i)$$

exists. This proves Proposition 3.

Corollary. $D\mathfrak{n}$ is a reflexive functor.

The mapping of functors $\kappa : F \to D^2F$ gives rise to mappings of functors $DF \to D^3F$ and $D^3F \to DF$, whose composition $DF \to D^3F \to DF$ is the identity mapping. For $F = \mathfrak{n}$, as we have shown, κ is an epimorphism; therefore $D^3\mathfrak{n} \to D\mathfrak{n}$ is a monomorphism, and consequently $D\mathfrak{n} = D^3\mathfrak{n}$.

Proposition 4. The spaces $\mathfrak{n}(A)$ and $D^2\mathfrak{n}(A)$ are topologically isomorphic if A is a reflexive ($A = \bar{\bar{A}}$) separable space in the extremal topology.

We outline the proof of this proposition. It is easy to see that in this case $D^2\mathfrak{n}(A) = H(D\mathfrak{n}(\bar{A}), R)$ and $D\mathfrak{n}(\bar{A}) \subseteq \mathfrak{n}(\bar{A})$, while $\mathfrak{n}(A)$ is a reflexive separable space in the extremal topology. Then every bicompact set in $\mathfrak{n}(A)$ is metrizable (7), and one can show that the extremal topology of the space $\overline{\mathfrak{n}(\bar{A})}$ induces on $D\mathfrak{n}(\bar{A})$ the extremal topology of the space $D\mathfrak{n}(\bar{A})$; hence, using the description of $\bar{X}$, we obtain that $\overline{D\mathfrak{n}(\bar{A})} = \overline{\mathfrak{n}(\bar{A})}$, i.e. $\overline{D^2\mathfrak{n}(A)} = \overline{\mathfrak{n}(A)}$, but $D^2\mathfrak{n}(A) \subseteq D^2\mathfrak{n}(A) = \mathfrak{n}(A)$. The latter shows that the algebraic isomorphism $\mathfrak{n}(A) = D^2\mathfrak{n}(A)$ is a topological isomorphism.

Remark. For arbitrary A we have only been able to show that the linear mapping $D^2\mathfrak{n}(A) \to \mathfrak{n}(A)$ is sequentially continuous.

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