ON THE CONSTRUCTION OF SOLUTIONS OF THE SYSTEM OF EQUATIONS OF BOUNDARY-LAYER THEORY BY THE METHOD OF LINES

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Abstract

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ON THE CONSTRUCTION OF SOLUTIONS OF THE SYSTEM OF EQUATIONS OF BOUNDARY-LAYER THEORY BY THE METHOD OF LINES

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The question of the existence and uniqueness of solutions of the Prandtl system for two-dimensional stationary and nonstationary flows of an incompressible fluid was studied in works ([1,2]). Below, solutions of the Prandtl system are constructed for axisymmetric three-dimensional nonstationary flows and two-dimensional flows by means of the method of lines, which is a variant of the well-known method of integral relations of A. A. Dorodnitsyn ([3,4]).

1. The Prandtl system for axisymmetric three-dimensional nonstationary flows of an incompressible fluid in the corresponding coordinates has the form (see ([5]), p. 174)

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \qquad (ru)_x + (rv)_y = 0 \tag{1}$$

in the domain $D\{0 \leq t \leq t_0,\ 0 \leq x \leq x_0,\ 0 \leq y < \infty\}$,

$$u|_{t=0} = u_0(x, y), \qquad u|_{x=0} = 0, \qquad u|_{y=0} = 0, \qquad v|_{y=0} = v_0(t, x),$$

$$u \to U(t, x) \quad \text{as } y \to \infty. \tag{2}$$

The function $r(x)$ determines the surface of the body being flowed around, $r(0) = 0$, $r(x) > 0$ for $x > 0$. By Bernoulli' s law, $-p_x = U_t + UU_x$; $U(t, 0) = 0$, $U(t, x) > 0$ for $x > 0$.

If one introduces new independent variables

$$\eta = u(t, x, y)/U(t, x), \qquad \xi = x, \qquad \tau = t, \tag{3}$$

then for the function $w = u_y/U$ we obtain the equation

$$\nu w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w = 0 \tag{4}$$

in the domain $\Omega\{0 \le \tau \le t_0,\ 0 \le \xi \le x_0,\ 0 \le \eta \le 1\}$ with the conditions

$$w|_{t=0} = u_{0y} = w_0(\xi,\eta), \quad w|_{\eta=1} = 0, \quad (\nu w w_\eta - v_0 w + C)|_{\eta=0} = 0, \tag{5}$$

where

$$A \equiv (\eta^2 - 1)U_x + (\eta - 1)U_{tU}^{-1}, \qquad B \equiv \eta r' r^{-1} U - \eta U_x - U_{tU}^{-1},$$

$$C \equiv U_x + U_{tU}^{-1}.$$

Using the method of lines, we shall prove, under appropriate assumptions, the existence and uniqueness of the solution of problem (4), (5), and then, as a consequence, obtain theorems on the existence and uniqueness of the solution of the Prandtl system (1), (2).

Let $f^{m,l} \equiv f(mh, lh, \eta)$. We replace equation (4) with conditions (5) by a system of ordinary differential equations

$$\nu(w^{m-1,l} + h)^2 w_{\eta\eta}^{m,l} - (w^{m,l} - w^{m-1,l})/h$$

$$-(\eta + h)U^{m,l}(w^{m,l} - w^{m,l-1})/h + A^{m,l} w_\eta^{m,l} + B^{m,l} w^{m,l} = 0 \tag{6}$$

with the conditions

$$w^{m,l}(1) = 0, \qquad \nu w^{m-1,l}(0) w_\eta^{m,l}(0) - v_0^{m,l} w^{m-1,l}(0) + C^{m,l} = 0, \tag{7}$$

$$m = 1, 2, \dots, \qquad l = 0, 1, 2, \dots; \qquad w^{0,l} \equiv w_0(lh, \eta), \qquad h = \text{const} > 0.$$

The solution of (6), (7) is reduced to the successive solution of linear second-order equations with prescribed boundary conditions for $m = 1$, $l = 0, 1, 2, \dots$, then $m = 2$, $l = 0, 1, 2, \dots$, etc. The solution of problem (6), (7) obviously exists if $w^{m-1,l}(0) \neq 0$ and $w^{m-1,l}(\eta) \geq 0$.

Lemma 1. *Let A, B, C, v_0 be bounded functions in Ω, let w_0 be continuous, and let $K_1(1-\eta) \le w_0(\xi,\eta) \le K_2(1-\eta)$, $K_1 > 0$, $K_2 > 0$, be constants. Then the solution of (6), (7) exists for $mh \le \tau_0$, where τ_0 is a certain number depending on the data of problem (1)-(2), and the estimate*

$$V(mh,\eta) \le w^{m,l}(\eta) \le V_1(mh,\eta)$$

holds, where the functions V *and* V_1*, continuous in* Ω*, are positive for* $\eta < 1$, $V = K_3(1-\eta)$, $V_1 = K_4(1-\eta)$ *in a neighborhood of* $\eta = 1$*, and* $K_3 > 0$, $K_4 > 0$ *are constants.*

Lemma 2. *Let the conditions of Lemma 1 be satisfied; let* A, B, C, v_0, w_0 *have bounded first derivatives,*

$$|w_{0\xi}| \leq K_5(1-\eta),$$

$w_0 w_{0\eta\eta}$ *be bounded in* Ω*, and let the compatibility condition*

$$\nu w_0 w_{0\eta} - v_0 w_0 + C = 0$$

hold for $\tau = 0$, $\eta = 0$*. Then*

$$w_\eta^{m,l}, \qquad (w^{m,l} - w^{m-1,l})/h, \qquad (w^{m,l} - w^{m,l-1})/h, \qquad (1-\eta+h)w_{\eta\eta}^{m,l}$$

are bounded in Ω *for* $mh < \tau_1$ *by constants independent of* h*;* τ_1 *is a certain number determined by the data of problem (1)-(2).*

Theorem 1. *Let the assumptions of Lemmas 1 and 2 be fulfilled. Then in the domain* Ω*, for* $\tau \leq \tau_1$*, there exists a solution* w *of problem (4), (5) possessing the following properties:* w *is continuous in* Ω *and*

$$K_6(1-\eta) \leq w \leq K_7(1-\eta),$$

$K_i = \text{const} > 0$*;* w *has bounded generalized derivatives* w_η, w_τ, w_ξ*; the derivative* w_η *is continuous with respect to* η *for* $\eta < 1$*,* w *satisfies the conditions (5); there exists a generalized derivative* $w_{\eta\eta}$ *such that* $(1-\eta)w_{\eta\eta}$ *is bounded in* Ω*; the function* w *satisfies equation (4) almost everywhere in* Ω*. The solution* w *of problem (4), (5) possessing the indicated properties is unique.*

The uniqueness of the solution w of problem (4), (5) follows from the energy inequality that holds for the equation satisfied by the difference of two solutions.

The functions $w^{m,l}(\eta)$, correspondingly extended linearly in τ and ξ in Ω, according to Lemmas 1, 2 form a compact family for $\tau \leq \tau_1$. Some subsequence of these functions converges uniformly to a function w as $h \to 0$. From the uniqueness of the limiting function w it follows that the whole family $w^{m,l}$ converges as $h \to 0$.

Theorem 2. *Let* $U_x, U_{tU}^{-1}, r'r^{-1}U, v_0$ *have bounded derivatives with respect to* t *and* x *in* D*,* $u_0 \to U$ *as* $y \to \infty$*,* $u_{0y} > 0$*,*

$$M_1(U-u_0) \leq u_{0y} \leq M_2(U-u_0),$$

$M_1 > 0$, $M_2 > 0$ *constants; let there exist bounded derivatives* u_{0yy}, u_{0yyy}, u_{0x}, u_{0xy}*, let the quantities* u_0/U, u_{0yy}/u_{0y}*, and*

$$(u_{0yyy}u_{0y} - u_{0yy}^2)/u_{0y}^2$$

be bounded and, moreover,

$$\left|(u_{0yx} - u_{0x}u_{0yy}/u_{0y}) + U_{xU}^{-1}(u_0 u_{0yy} - u_{0y}^2)/u_{0y}\right| \leq M_3(U-u_0).$$

Let the compatibility condition

$$v_0(0,x)u_{0y}(x,0) = -p_x(0,x) + \nu u_{0yy}(x,0) \tag{8}$$

be fulfilled. Then in D there exists, uniquely for $t \leq \tau_1$, a solution u, v of problem (1), (2) possessing the following properties: u, u_y are continuous and bounded, $u_y > 0$ for $y \geq 0$, $u_y \to 0$ as $y \to \infty$, v is continuous in y and bounded for bounded y. There exist bounded generalized derivatives u_t, u_x, u_{yy}, v_y, and the system (1) is satisfied almost everywhere; in addition, the following quantities are bounded:

$$u_y/U, \qquad u_{yy}/u_y, \qquad (u_{yyy}u_y - u_{yy}^2)/u_y^2,$$

$$U^{-1}(u_{yx}-u_{xu_{yy}}/u_y)+U_x(uu_{yy}-u_y^2)/u_{yU}^2, \qquad U^{-1}(u_{yt}-u_{tu_{yy}}/u_y)+U_t(uu_{yy}-u_y^2)/u_{yU}^2.$$

In order to obtain a smoother solution of problem (1), (2), we must consider another system of ordinary differential equations approximating (4), (5). By $f^{m,l,n}$ we shall denote

value of the function $f_n(\tau,\xi,\eta)$ at the point (mh, lh, η). For $0 \leq \eta \leq 1$ consider the system

$$\nu \left(w^{m,l,n-1}\right)^2 w_{\eta\eta}^{m,l,n} - \left(w^{m,l,n} - w^{m-1,l,n}\right)/h$$
$$-(\eta+h)U^{m,l}\left(w^{m,l,n} - w^{m,l-1,n}\right)/h + A^{m,l}w_\eta^{m,l,n} + B^{m,l}w^{m,l,n} = 0 \tag{9}$$

with the conditions

$$w^{m,l,n}(1) = 0, \qquad \nu w^{m,l,n-1}(0)w_\eta^{m,l,n}(0) - v_0^{m,l}w^{m,l,n-1}(0) + C^{m,l} = 0, \tag{10}$$

$$n, m = 1, 2, \dots, \qquad l = 0, 1, 2, \dots, \qquad w^{0,l,n} = w_0(lh,\eta),$$
$$w^{m,l,0} = w_0(lh,\eta).$$

To prove the existence of a solution of (9), (10) with continuous derivatives of fourth order for $0 \leq \eta \leq 1$ for $mh \leq \tau_2$, we use the method of introducing a small parameter and enlarging the domain, as was done in constructing smooth solutions in papers ([2],[6]).

Lemma 3. *Let the conditions of Lemmas 1 and 2 be satisfied; A, B, C, v_0, w have bounded derivatives of second order; $w_0^2 w_{0\eta\xi}$, $w_0^2 w_{0\eta\eta\eta}$, $w_0^2 w_{0\eta\eta\eta\eta}$ are bounded, and suppose the compatibility condition*

$$\nu w_0 w_\tau + \nu w_{0\eta} w_\tau - v_{0\tau} w_0 - v_0 w_\tau + C_\tau = 0$$

is satisfied for $\eta = 0, \tau = 0$, where

$$w_\tau \equiv \nu w_0^2 w_{0\eta\eta} + A w_{0\eta} + B w_0.$$

Then in Ω, for $\tau \leq \tau_3$, the quantities

$$w^{m,l,n}, \quad w_\eta^{m,l,n}, \quad \left(w^{m,l,n} - w^{m-1,l,n}\right)/h,$$

$$\left(w^{m,l,n} - w^{m,l-1,n}\right)/h, \quad w_{\eta\eta}^{m,l,n}, \quad \left(w^{m+1,l,n} - 2w^{m,l,n} + w^{m-1,l,n}\right)/h^2,$$

$$\left(w_\eta^{m,l,n} - w_\eta^{m-1,l,n}\right)/h, \quad \left(w_\eta^{m,l,n} - w_\eta^{m,l-1,n}\right)/h,$$

$$\left(w^{m,l+1,n} - 2w^{m,l,n} + w^{m,l-1,n}\right)/h^2,$$

$$\left(w^{m,l,n} - w^{m,l-1,n} - w^{m-1,l,n} + w^{m-1,l-1,n}\right)/h^2$$

are bounded by constants independent of n and h; $\tau_3 > 0$ and is determined by the data of problem (1), (2).

It follows from Lemma 3 that the solution w of problem (4), (5), whose existence is asserted in Theorem 1, has, for $\tau < \tau_3$, under the assumptions of Lemma 3, first-order derivatives satisfying the Lipschitz condition and a bounded derivative $w_{\eta\eta}$, continuous for $\eta < 1$. Hence we obtain the following assertion:

Theorem 3. *Let the conditions of Theorem 2 be satisfied; U_x, U_{tU}^{-1}, $r'r^{-1}U$, v_0 have bounded derivatives of second order, and the initial function w_0 be such that for $w_{0y} \equiv w_0(\xi, \eta)$ the smoothness conditions and the compatibility conditions of Lemma 3 are satisfied. Then the solution u, v of problem (1), (2), whose existence is asserted in Theorem 2, has in D, for $t \leq \tau_3$, continuous and bounded derivatives entering system (1).*

We note that Theorems 1, 2, 3 also hold for symmetric two-dimensional flows. The boundary-layer equations for such flows, as is known, have the form of system (1) with $r(x) \equiv 1$ and with conditions (2) on the boundary of the domain D.

2. By the method of lines, exactly as was done above for axisymmetric flows, one can prove the existence of solutions of the Prandtl system for two-dimensional flows and obtain theorems analogous to those obtained earlier in papers ([1],[2]). Consider system (1) for $r(x) > 0$ for $x \geq 0$, or $r(x) \equiv 1$, with the conditions

$$u\big|_{t=0} = u_0(x, y), \qquad u\big|_{x=0} = u_1(t, y), \qquad u\big|_{y=0} = 0, \qquad v\big|_{y=0} = v_0(t, x),$$

$$u \to U(t, x) \quad \text{as } y \to \infty. \tag{11}$$

We assume that $U(t, x) > 0$ for $x \geq 0$, $u_0 > 0$, $u_1 > 0$ for $y > 0$. The change of independent variables (3) leads, for the function $w = u_y/U$, to equation (4), conditions (5), and the condition

$$w\big|_{\xi=0} = u_{1y} \equiv w_1(\tau, \eta). \tag{12}$$

Consider system (6) for $m = 1, 2, \ldots,\ l = 1, 2, \ldots$ and adjoin to conditions (7) the condition $w^{m,0} = w_1(mh, \eta)$. For solutions of such a system, under the corresponding assumptions on w_1, lemmas analogous to Lemmas 1 and 2 are valid, which leads to the following theorem.

Theorem 4. *Let the assumptions of Lemmas 1 and 2 be satisfied with respect to U, v_0, w_0. Let $K_8(1-\eta) \le w_1 \le K_9(1-\eta)$, and suppose there exist bounded*

continuous derivatives $w_{1\eta}, w_{1\tau}, w_1 w_{1\eta\eta}$, with $|w_{1\tau}| \le K_{10}(1-\eta)$, $K_i > 0$ constants. Suppose that the compatibility conditions are satisfied: $w_0 = w_1$ for $\tau = 0, \xi = 0$, $\nu w_1 w_{1\eta} - v_0 w_1 + C = 0$ and $\nu w_1^2 w_{1\eta\eta} - w_{1\tau} + A w_{1\eta} + B w_1 = 0$ for $\eta = 0, \xi = 0$. Then in the domain Ω, if $\tau \le \tau^1$ or if $\xi \le \xi^1$, there exists a unique solution w of problem (4), (5), (12), having the same properties as the solution w indicated in Theorem 1; τ^1 and ξ^1 are positive constants determined by the data of problem (1), (11).

Theorem 5. Suppose $U(x) > 0$, and U_x, U_t, v_0 have bounded first derivatives, $u_0 \to U$, $u_1 \to U$ as $y \to \infty$, $u_{0y} > 0$, $u_{1y} > 0$ for $y \ge 0$; $M_4(U - u_0) \le u_{0y} \le M_5(U - u_0)$, $M_6(U - u_1) \le u_{1y} \le M_7(U - u_0)$, where M_i are positive constants; for u_0 and u_1 there exist bounded derivatives u_{yy}, u_{yyy}, and also $u_{0x}, u_{0xy}, u_{1t}, u_{1ty}$; for u_0 and u_1 the quantities u_{yy}/u_y, $(u_{yyy}u_y - u_{yy}^2)/u_y^2$ are bounded; $|w_{0\xi}| \le M_8(U - u_0)$, $|w_{1\tau}| \le M_9(U - u_1)$. Suppose that the compatibility condition (8) is satisfied; $u_0 = u_1$ for $t = 0, x = 0$; $u_1 = 0$, $u_0 = 0$ for $y = 0$, and, moreover, $v_0 u_{1y} = -p_x + \nu u_{1yy}$, $u_{1t} + v_0 u_{1y} = \nu u_{1yyy}$ for $y = 0, x = 0$. Then in the domain D, for $t_0 \le \tau^1$ or $x_0 \le \xi^1$, there exists a unique solution of problem (1), (11), having the properties indicated in Theorem 2.

Consider the system

$$u u_x + v u_y = -p_x + \nu u_{yy}, \qquad u_x + v_y = 0 \tag{13}$$

in the domain $D_1\{0 \le x \le x_1,\ 0 \le y < \infty\}$ with the conditions

$$u\big|_{x=0} = u_1(y), \quad u\big|_{y=0} = 0, \quad v\big|_{y=0} = v_0(x), \quad u \to U(x) \quad \text{as } y \to \infty. \tag{14}$$

We shall assume that $U(x) > 0$ for $x \ge 0$, $u_1 > 0$ for $y > 0$, $u_{1y} > 0$ for $y \ge 0$. The change of independent variables (3) and the introduction of the function $w = u_y/U$ reduce problem (13), (14) to the equation

$$\nu w^2 w_{\eta\eta} - \eta U w_\xi + (\eta^2 - 1)U_{xww_\eta} - \eta U_{xw} = 0 \tag{15}$$

in the domain $\Omega_1\{0 \le \xi \le x_1,\ 0 \le \eta \le 1\}$ with the conditions

$$w\big|_{\xi=0} = u_{1y} \equiv w_1(\eta), \quad w\big|_{\eta=1} = 0, \quad (\nu w w_\eta - v_0 w + U_x)\big|_{\eta=0} = 0. \tag{16}$$

Let $f^l(\eta) \equiv f(lh, \eta)$. The system of ordinary differential equations

$$\nu(w^{l-1}+h)^2 w^l_{\eta\eta} - (\eta+h)U(w^l - w^{l-1})/h + (\eta^2-1)U^l_{xw_\eta} - \eta U^l_{xw} = 0, \quad (17)$$

$l = 1, 2, \dots,\ w^0(\eta) = w_1(\eta)$, with the conditions

$$w^l(1) = 0, \quad \nu w^{l-1} w^l_\eta - v_0 w^{l-1} + U_x = 0 \quad \text{for } \eta = 0$$

has the same properties as the system constructed for problem (4), (12), (5). As $h \to 0$, $w^l(\eta)$ converge to the solution of problem (15), (16), from the solvability of which we obtain a theorem on the existence of a solution of problem (13), (14).

Theorem 6. Suppose U_x, v_0 have bounded first derivatives, and $u_1(y)$ satisfies the smoothness conditions and the compatibility conditions indicated in Theorem 5. Then in the domain D_1, for $x \le x^1$, there exists a unique solution u, v of problem (13), (14), having the properties indicated in Theorem 2. Moreover, all derivatives entering into system (13) are continuous at interior points of the domain; x^1 depends on the data of the problem.

For solutions of problems (13), (14) and (1), (11), theorems analogous to Theorem 3 are also valid.

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