SOME QUESTIONS ON APPROXIMATION OF FUNCTIONS BY ALGEBRAIC POLYNOMIALS IN THE INTEGRAL METRIC

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Abstract

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SOME QUESTIONS ON APPROXIMATION OF FUNCTIONS BY ALGEBRAIC POLYNOMIALS IN THE INTEGRAL METRIC

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1. In the present paper we give some results connected with the question of a constructive characterization of nonperiodic functions in the integral metric. In the metric C this question was completely solved in the works of S. M. Nikol' skii (^{1}), A. F. Timan (^{2,3}), and V. K. Dzyadyk (^{4}). In ($^{2-4}$) it was established that, in order that a function $f(x)$, given on the segment $[-1,1]$, have there, for some nonnegative integer r, a derivative $f^r(x)$ of order r belonging to the class $\mathrm{Lip}\,\alpha$ $(\alpha < 1)$, it is necessary and sufficient that for every natural n there be an ordinary polynomial $P_n(x)$ of degree not exceeding n such that, for all $x \in [-1,1]$, the inequality

$$|f(x) - P_n(x)| \le C\left(\sqrt{1-x^2}/n + 1/n^2\right)^{r+\alpha}.$$

   holds.

In the integral metric the first works in this direction were carried out by G. K. Lebed' and M. K. Potapov (^{5,6}).

2. In what follows we shall everywhere consider classes of functions integrable to the p-th power with norm

$$\|f\|_p = \left\{\int_{-1}^{1} |f(x)|^p\,dx\right\}^{1/p}.$$

   By $W^{(r)}H_p^\omega$ (respectively $W^{(r)}A_p^\omega$), $1 \le p < \infty$, we denote the class of functions given on the interval $[-1,1]$ and having there an r-th derivative $f^{(r)}(x)$, Lebesgue integrable to the p-th power, for which, for any $h > 0$, the inequality

$$\|f^{(r)}(x+h) - f^{(r)}(x)\|_{L_p(-1,\,1-h)} \le \omega(h)$$

holds, respectively the inequality

$$\left\|\frac{f^{(r)}\left(x\sqrt{1-h^2}-h\sqrt{1-x^2}\right)-f^{(r)}(x)}{\omega\left(\sqrt{1-x^2}\,h+h^2\right)}\right\|_{L_p}\le C.$$

For $\omega(t)=t^\alpha$ these classes will be denoted by $H_p^{(r+\alpha)}$ (respectively $A_p^{(r+\alpha)}$). The classes $W^{(r)}A_p^\omega$ were introduced by G. K. Lebed' (^{5}) and M. K. Potapov (^{6,7}). It is known that for $p=\infty$ the classes $W^{(r)}H_\infty^\omega$ and $W^{(r)}A_\infty^\omega$ coincide (see, for example, (^{5})), while for $1\le p<\infty$ the intersection of these classes is nonempty.

G. K. Lebed'and M. K. Potapov gave the following constructive characterization of the class $A_p^{(r+\alpha)}$ for $0<\alpha<1$. In order that $f(x)\in A_p^{(r+\alpha)}$, it is necessary and sufficient that for every natural number $n\ge 0$ there be an algebraic polynomial of degree not exceeding n such that the inequality

$$\left\|\frac{f(x)-P_n(x)}{\left(\sqrt{1-x^2}+1/n\right)^{r+\alpha}}\right\|_{L_p}<\frac{C}{n^{r+\alpha}},$$

holds, where the constant C does not depend on n and $f(x)$. However, even after these works there remained open the question of a constructive characterization of the class $W^{(r)}H_p^\omega$, and also the connection between the classes $W^{(r)}H_p^\omega$ and $W^{(r)}A_p^\omega$ had not been clarified.

3. In this section we formulate several theorems showing how the classes $W^{(r)}H_p^\omega$ and $W^{(r)}A_p^\omega$ are related. Obviously, it is enough for us to consider the case $r=0$.

Theorem 1. For any function $f(x)\in W^{(0)}H_p^\omega$ the inequality

$$\left\|\frac{f\left(x\sqrt{1-h^2}-h\sqrt{1-x^2}\right)-f(x)}{\omega\left(\sqrt{1-x^2}\,h+h^2\right)}\right\|_{L_p}\le C\ln^{1/p}\frac{1}{h}. \tag{1}$$

Theorem 2. For any function $f(x)\in W^{(0)}A_p^\omega$ the inequality

$$\|f(x+h)-f(x)\|_{L_p(-1,\,1-h)}\le C\omega(h)\ln^{1/p}h\frac{1}{h}. \tag{2}$$

It can be shown by examples that, in the case $\omega(t)=t^\alpha$ $(0<\alpha<1)$, there exist functions $f_h(x)$ and $\bar{f}_h(x)$, belonging respectively to the classes H_p^α and A_p^α, and a constant C_0 (C_0, generally speaking, depends on α) such that the sign of the inequality in (1) and (2) is reversed. In some cases the logarithms in the right-hand side of inequalities (1) and (2) can be omitted. For example, the classes of functions H_p^1 and A_p^1 coincide (consequently, the classes of functions $H_p^{(r+1)}$ and A_p^{r+1} coincide for any integer r). Besides this case, it is interesting to note one more case of coincidence of the classes H_p^α and A_p^α. A function $f(x)$, defined on $[-1,1]$, belongs to the intersection of the classes H_p^α and A_p^α if it satisfies the condition

$$\sum_i |f(x_i)-f(x_{i+1})|^p |x_i - x_{i+1}|^{-\alpha p+1} = O(1), \tag{3}$$

where x_i are arbitrary points of the interval $[-1,1]$ $(x_{i+1} > x_i)$; moreover, it is sufficient that condition (3) hold not on the whole interval, but only near the endpoints.

4. It follows from Theorem 1 that if $f(x) \in H_p^{(r+\alpha)}$, then for each $n = 0, 1, 2, \ldots$ there exists an algebraic polynomial of degree not exceeding n such that the inequality

   $$\left\| \frac{f(x)-P_n(x)}{\left(\sqrt{1-x^2}+1/n\right)^{r+\alpha}} \right\|_{L_p} \le C \frac{\ln^{1/p} n}{n^{r+\alpha}}. \tag{4}$$

   holds.

This result is one of the variants of the strengthened Jackson theorem in the metric L_p. However, inequality (4) does not characterize the class $H_p^{(r+\alpha)}$, since functions of the class $A_p^{(r+\alpha)}$ also satisfy this inequality. But, on the other hand, in the right-hand side of inequality (4) one cannot replace the quantity $\frac{1}{n^\alpha}\ln^{1/p} n$ by a function $\omega(1/n)$ such that

$$\overline{\lim_{h\to 0}} \left[h \int_h^1 \frac{\omega(u)}{u^2}\, du \right]^{-1} h^\alpha \ln^{1/p} \frac{1}{h} = \infty. \tag{5}$$

The latter follows from the inverse theorems of G. K. Lebed’ (5) and the results of Sec. 3. As a consequence of inequality (4), we note Theorem 3.

Theorem 3. Any function $f(x)$, belonging on the segment $[-1,1]$ to the class $H_p^{(r+\alpha)}$ $(r+\alpha > 1/4)$, is expanded in a Fourier-Legendre series converging in the metric L to the function $f(x)$, and the estimate

$$\int_{-1}^{1} \left| f(x) - \sum_{k=0}^{n} a_k P_k(x) \right| dx \le C \begin{cases} \ln n / n^{2\alpha - 1/2}, & r+\alpha < 1/2, \\ \ln n / n^{r+\alpha}, & r+\alpha > 1/2. \end{cases} \tag{6}$$

holds.

It was indicated above (see Sec. 2) that the class A_p^α, in terms of approximation by algebraic polynomials, can be defined as the class of functions for which the inequality

$$\left\| \frac{f(x)-P_n(x)}{\rho_0^\alpha (1-x^2+1/n^2)} \right\|_{L_p} = O\left(\frac{1}{n^\alpha}\right), \tag{7}$$

holds.

where $P_n(x)$ is an algebraic polynomial of degree not exceeding n, $\rho_0(t) = \sqrt{t}$. Naturally, the question arises: can $\rho(t)$ be determined so that equality (7) would be equivalent to the assertion that $f(x) \in H_p^\omega$? The answer to this question is negative.

The following variant of an strengthened Jackson theorem in the integral metric is proposed.

Theorem 4. *If a function $f(x)$ is given on the interval $[-1, 1]$ and belongs to the class $W^{(0)}H_p^\omega$, then there exists a sequence of algebraic polynomials $P_n(x)$ such that, for every n, the inequality*

$$\left\{ \sum_{i=-k-1}^{k} \int_{a_i}^{a_{i+1}} |f(x) - P_{n_i}(x)|^p \, dx \right\}^{1/p} = O\left[\omega\left(\frac{1}{n}\right)\right], \tag{8}$$

holds, where

$$n_i = [n\sqrt{1 - a_i}];$$

a_i are points of the interval $[-1, 1]$ such that

$$a_{-i} = a_i, \qquad \sqrt{1 - a_i} = \frac{1}{2^i}, \qquad 2\sqrt{n} \geq 2^k \geq \sqrt{n}, \qquad a_{k+1} = 1 \quad (i = 0, 1, \dots, k).$$

There is also a theorem converse to Theorem 4.

Theorem 5. *If, for a function $f(x)$ given on the interval $[-1, 1]$, there exists a sequence of algebraic polynomials such that inequality (8) holds, where $\omega(t)$ is some modulus of continuity, then*

$$\omega(f; t)_{L_p} = O\left[t \int_t^1 \frac{\omega(u)}{u^2} \ln \frac{2u}{t} \, du\right].$$

Thus, a theorem has been proved which gives a constructive characterization of the class $W^{(0)}H_p^\omega$ under the additional assumption that $\omega(t)$ satisfies the condition

$$t \int_t^1 \frac{\omega(u)}{u^2} \, du = O[\omega(t)]. \tag{9}$$

Theorem 6. *In order that a function $f(x)$, given on the interval $[-1, 1]$, belong to the class $W^{(0)}H_p^\omega$, where $\omega(t)$ satisfies condition (9), it is necessary and sufficient that inequality (8) hold.*

Remark. The points a_i may be chosen arbitrarily, but so that

$$\sqrt{1 - a_i} < A\sqrt{1 - a_{i+1}},$$

where A is some positive number. In this case, of course, the constant in the right-hand side of inequality (8) depends on A.

For classes of functions $W^{(r)}H_p^\omega$ $(r > 0)$, under the condition that $\omega(t)$ satisfies condition (9) and

$$\int_0^t \frac{\omega(u)}{u}\,du = O[\omega(t)],$$

the following is valid.

Theorem 7. *In order that a function, given on the interval* $[-1,1]$, *belong to the class* $W^{(r)}H_p^\omega$, *it is necessary and sufficient that there exist a sequence of algebraic polynomials such that, for every* n, *the inequality*

$$\left\{\sum_{i=-k-1}^{k}\int_{a_i}^{a_{i+1}} |f(x) - P_{n_i}(x)|\,dx\right\}^{1/p} = O\left[\frac{1}{n^r}\omega\left(\frac{1}{n}\right)\right], \tag{10}$$

holds, where

$$n_i = [n\sqrt{1-a_i}],$$

a_i *are points of the interval* $[-1,1]$ *such that*

$$a_{-i} = a_i, \qquad \sqrt{1-a_i} = \frac{1}{2^i}, \qquad a_{k+1} = 1 \quad (i = 0, 1, \dots, k); \qquad 2\sqrt{n} \geq 2^k \geq \sqrt{n}.$$

5. In the works of M. K. Potapov (6,7) and G. K. Lebed' (5), other functional spaces L_p with norm $\|\bar{f}\|_{L_p(-1,1)} =$ were also considered.

$$= \left\{\int_{-1}^{1} |f(x)|^p \frac{dx}{\sqrt{1-x^2}}\right\}^{1/p}.$$

By definition, $f(x) \in W^{(r)}\widetilde{H}_p^\alpha$, if $f(x)$ has an r-th derivative $f^{(r)}(x)$ for which, for any $h > 0$, the inequality

$$\left\|\overline{f(x+h) - f(x)}\right\|_{L_p(-1,1-h)} < h^\alpha$$

holds, and $f(x) \in W^{(r)}\widetilde{A}_p^\alpha$, if the r-th derivative $f^{(r)}(x)$ exists and satisfies the condition

$$\left\|\frac{f^{(r)}(x\sqrt{1-h^2} - h\sqrt{1-x^2}) - f(x)}{\left(\sqrt{1-x^2}\,h + h^2\right)^\alpha}\right\|_{L_p(-1,1)} = O(1).$$

As M. K. Potapov showed (7), the definition of the class $W^{(r)}\widetilde{A}_p^\alpha$ in differential-difference terms is equivalent to the following definition of this class in terms of approximation by algebraic polynomials: the class $W^{(r)}\widetilde{A}_p^\alpha$ consists of those and only those functions $f(x)$, defined on the interval $[-1,1]$, for which

$$\left\| \frac{f(x) - P_n(x)}{\left(\sqrt{1-x^2} + 1/n\right)^{r+\alpha}} \right\|_{L_p(-1,1)} = O\left(\frac{1}{n^{r+\alpha}}\right). \tag{11}$$

In the same work M. K. Potapov posed the problem of a constructive characterization of the class $W^{(r)}\widetilde{H}_p^\alpha$ and of the relation between the classes $W^{(r)}\widetilde{H}_p^\alpha$ and $W^{(r)}\widetilde{A}_p^\alpha$. Of all the theorems analogous in meaning to Theorems 1-7, we note the following one.

Theorem 8. *In order that a function $f(x)$, defined on the interval $[-1,1]$, belong to the class $W^{(r)}\widetilde{H}_p^\alpha$, it is necessary and sufficient that there exist a sequence of polynomials $P_n(x)$ such that*

$$\left\{ \sum_i \int_{a_i}^{a_{i+1}} \left| f(x) - P_{n_i}(x) \right| \frac{dx}{\sqrt{1-x^2}} \right\}^{1/p} = O\left(\frac{1}{n^{r+\alpha}}\right), \tag{12}$$

where $n_i = [n\sqrt{1-a_i}]$ and $\sqrt{1-a_i} < A\sqrt{1-a_{i+1}}$, A is some fixed number.

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CITED LITERATURE

[1] S. M. Nikol' skii, *Izv. AN SSSR, ser. matem.*, **10**, No. 4 (1946).
[2] A. F. Timan, *DAN*, **77**, 969 (1951).
[3] A. F. Timan, *DAN*, **78**, 17 (1951).
[4] V. K. Dzyadyk, *Izv. AN SSSR, Ser. matem.*, **20**, 623 (1956).
[5] G. K. Lebed' , *DAN*, **118**, 239 (1958).
[6] M. K. Potapov, *DAN*, **111**, 1185 (1956).
[7] M. K. Potapov, *Vestn. Mosk. univ.*, No. 4 (1960).

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