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ON THE SOLUTION OF A DIFFUSION PROBLEM WITH A NONLINEAR BOUNDARY CONDITION

HYDROMECHANICS

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Abstract

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HYDROMECHANICS

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ON THE SOLUTION OF A DIFFUSION PROBLEM WITH A NONLINEAR BOUNDARY CONDITION

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The problem of self-oscillations arising under certain conditions in electrochemical systems with a falling characteristic is reduced to finding a periodic solution of the diffusion equation $\partial c/\partial t = D\,\partial^2 c/\partial x^2$ for the concentration c of the discharging substance in the semi-infinite region $0 \le x < \infty$ with the nonlinear boundary condition $\frac{dc(0,t)}{dx} = \chi_i[c(0,t)]$ ($i = 1$ for $0 < t < T_1$, $i = 2$ for $T_1 < t < T$), where D is the diffusion coefficient, $\chi(c)$ is an S-shaped function, $c(0,0) = c(0,T) = c_-$, $c(0,T_1) = c_+$, $c_- = \min c(0,t)$, $c_+ = \max c(0,t)$, and the value c_- is attained by the function $c(0,t)$ only at $t = 0$ and $t = T$, while the value c_+ only at $t = T_1$, the period T and the quantity T_1 being unknown in advance ([1],[2]). Below this problem is reduced to the solution of a nonlinear integral equation, and it is shown how to find this solution in some cases.

We shall seek the concentration in the form $c(x,t) = qx + r + u(x,t)$, where the quantities q and r are unknown in advance and $\lim_{x\to\infty} u(x,t) = 0$; then it is necessary to find a periodic solution of the equation

$$\partial u/\partial t = D\,\partial^2 u/\partial x^2 \tag{1}$$

with the nonlinear boundary condition

$$\begin{aligned} &\partial u(0,t)/\partial x = \Psi_i[u(0,t)] \quad (\Psi_i[u(0,t)] = \chi_i[c(0,t)] - q, \quad i = 1,2, \\ &u(0,0) = u(0,T) = u_-, \quad u(0,T_1) = u_+, \quad \Psi_1'(u_+) = \Psi_2'(u_-) = \infty), \end{aligned} \tag{2}$$

where $\Psi(u)$ is an S-shaped function (Fig. 1, solid curves).

Assuming at first $\Psi_i[u(0,t)]$ and $u(x,0) = \Phi(x)$ to be known functions of t and x, respectively, we write the solution of (1)–(2) as

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$$u(x,t) = \frac{1}{\sqrt{\pi D t}} \int_0^\infty \Phi(\alpha) K(x,\alpha,Dt)\, d\alpha - \sqrt{\frac{D}{\pi}} \int_0^t \frac{\Psi[u(0,\tau)] \exp[-x^2/4D(t-\tau)]\, d\tau}{\sqrt{t-\tau}} \tag{3}$$

$$(K(x,\alpha,Dt) = \exp[-x^2/4Dt - \alpha^2/4Dt] \operatorname{ch} x\alpha/2Dt).$$

Putting $t = T$ in formula (3) and taking into account that $u(x,T) = u(x,0) = \Phi(x)$, we reduce the problem to the solution of a linear integral equation for the function $\Phi(x)$, under the assumption that $\Psi_i[u(0,t)]$, T_1, and T are known:

$$\Phi(x) = \frac{1}{\sqrt{\pi D t}} \int_0^\infty \Phi(\alpha) K(x,\alpha,DT)\, d\alpha + \Omega(x) \tag{4}$$

$$\left(\Omega(x) = -\sqrt{\frac{D}{\pi}} \left\{ \int_{T_1}^T \frac{\Psi_2[u(0,\sigma)] \exp[-x^2/4D(T-\sigma)]\, d\sigma}{\sqrt{T-\sigma}} + \int_0^{T_1} \frac{\Psi_1[u(0,\sigma)] \exp[-x^2/4D(T-\sigma)]\, d\sigma}{\sqrt{T-\sigma}} \right\} \right).$$

The homogeneous equation (4) has the solution $\Phi(x) \equiv C$; the nonhomogeneous equation has a solution only if $\int_0^\infty \Omega(x)\, dx = 0$. Changing, in this condition, where the notation (4) has been introduced, the order of integration, we reduce it to the form

$$\int_{T_1}^T \Psi_2[u_2(0,\sigma)]\, d\sigma + \int_0^{T_1} \Psi_1[u_1(0,\sigma)]\, d\sigma = 0. \tag{5}$$

Using equality (5), by the method of successive approximations we find the solution $\Phi(x)$ of equation (4) in the form of a series uniformly convergent by Abel's test (3). Substituting this series into formula (3) and performing termwise integration, we obtain the functions $u_1(x,t)$ $(0 \le t \le T_1)$ and $u_2(x,t)$ $(T_1 \le t \le T)$ in the form of uniformly convergent series

$$u_1(x,t) = -\sqrt{\frac{D}{\pi}} \left\{ \int_0^t \exp\left[-\frac{x^2}{4D(t-\sigma)}\right] \frac{\Psi_1[u_1(0,\sigma)]\, d\sigma}{\sqrt{t-\sigma}} + S(x,t) \right\},$$

$$\begin{aligned} u_2(x,t) = -\sqrt{\frac{D}{\pi}} \Bigg\{ &\int_{T_1}^t \exp\left[-\frac{x^2}{4D(t-\sigma)}\right] \frac{\Psi_2[u_2(0,\sigma)]\, d\sigma}{\sqrt{t-\sigma}} \\ &+ \int_0^{T_1} \exp\left[-\frac{x^2}{4D(t-\sigma)}\right] \frac{\Psi_1[u_1(0,\sigma)]\, d\sigma}{\sqrt{t-\sigma}} + S(x,t) \Bigg\}, \end{aligned} \tag{6}$$

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$$S(x,t) = \sum_{n=1}^{\infty} \left\{ \int_{T_1}^{T} \exp\left[-\frac{x^2}{4D(nT+t-\sigma)}\right] \frac{\Psi_2[u_2(0,\sigma)]\,d\sigma}{\sqrt{nT+t-\sigma}} + \int_0^{T_1} \exp\left[-\frac{x^2}{4D(nT+t-\sigma)}\right] \frac{\Psi_1[u_1(0,\sigma)]\,d\sigma}{\sqrt{nT+t-\sigma}} \right\}.$$

Putting $x = 0$ in formulas (6), we reduce the problem under consideration to finding a solution of a nonlinear integral equation for determining, under fulfillment of condition (5), the functions $u_i(0,t)$ $(i = 1,2)$ and the quantities T_1, T, q, and r.

Suppose that $T_1 = T/2$, $\Psi_2[u_2(0,T/2+\tau)] \equiv -\Psi_1[u_1(0,\tau)]$. From solution (6) it is easy to see that then $u_2(0,T/2+\tau) = -u_1(0,\tau)$, $\Psi_2(u) = -\Psi_1(-u)$. Returning to the originally formulated problem for the concentration $c(x,t)$, we find that in this case (which we shall call symmetric), denoting $u_1(0,t) = u(t)$, $\Psi_1(u_1) = \Psi(u)$, $r = (c_+ + c_-)/2$, $\Psi_i(v) = \chi_i[(c_+ + c_-)/2 + v] - q$ $(i = 1,2)$.

The series with general term

$$v_n(\sigma) = \Psi[u(\sigma)]\left\{-[(n-1/2)T + \tau - \sigma]^{-1/2} + [nT + \tau - \sigma]^{-1/2}\right\}$$

converges uniformly in the interval $0 \le \sigma \le T/2$, and the functions v_n are integrable; consequently, (6), where $x = 0$ has been put, reduces to the equation for finding the function $u(t)$

$$u(t) = -\sqrt{\frac{D}{\pi}} \left\{ \int_0^t \frac{\Psi[u(\sigma)]\,d\sigma}{\sqrt{t-\sigma}} + \int_0^{T/2} \frac{\Psi[u(\sigma)]}{\sqrt{T}}\, Q\left(\frac{t-\sigma}{T}\right) d\sigma \right\} \qquad \left(0 \le t \le \frac{T}{2}\right) \tag{7}$$

$$\left(Q(z) = \sum_{n=1}^{\infty} \left\{ -\frac{1}{\sqrt{n - \frac{1}{2} + z}} + \frac{1}{\sqrt{n+z}} \right\} \right).$$

Putting $U = u/u_+$, $\tau = t/T$, $\lambda = \mu\sqrt{DT}/u_+$ and assuming that condition (2) is written in the form $\partial u(0,t)/\partial x = \mu F(U)$, we rewrite (7) in the form

$$U(\tau) = -\frac{\lambda}{\sqrt{\pi}} \left\{ \int_0^{\tau} \frac{F[U(\sigma)]\,d\sigma}{\sqrt{\tau-\sigma}} + \int_0^{1/2} F[U(\sigma)]\,Q(\tau-\sigma)\,d\sigma \right\}. \tag{8}$$

We shall consider the space C of continuous functions. Let $F(U) = F_0(U) + \varepsilon\xi(U)$, where $F_0(U)$ is a function for which the solution $\lambda\overline{U}(\tau)$ of equation (8) is known, $\varepsilon > 0$ is a small parameter, and $\xi(U)$ is a function satisfying the Lipschitz condition

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$$\|\xi(U'') - \xi(U')\| < L\|U'' - U'\|. \tag{9}$$

Denoting by BU the operator

$$BU = \frac{1}{\sqrt{\pi}} \left\{ \int_0^{\tau} \frac{\xi[U(\sigma)]\, d\sigma}{\sqrt{\tau - \sigma}} + \int_0^{1/2} \xi[U(\sigma)] Q(\tau - \sigma)\, d\sigma \right\}, \tag{10}$$

we reduce equation (8) to the form

$$U = \lambda \overline{U} - \varepsilon \lambda BU, \tag{11}$$

where (9) shows that the operator BU satisfies the Lipschitz condition

$$\|BU'' - BU'\| < a\|U'' - U'\|. \tag{12}$$

We shall solve equation (11) by the method of successive approximations, putting

$$U_{n+1} = \lambda_{n+1}\overline{U} - \varepsilon\lambda_n BU_n. \tag{13}$$

We shall consider the functions $U_n(\tau)$ $(0 \le \tau \le 1/2)$. As is easy to see, $\overline{U}(0) = -\overline{U}(1/2)$, $BU_n(0) = -BU_n(1/2)$, and consequently, from (13), $U_{n+1}(0) = -U_{n+1}(1/2)$. The value λ_{n+1} is determined from the condition $U_{n+1}(1/2) = 1$:

$$\lambda_{n+1} = [1 + \varepsilon\lambda_n BU_n(1/2)]/\overline{U}(1/2). \tag{14}$$

From (13) it is clear that all the functions $U_n(\tau)$ are continuous. Since for $\tau = 1/2$ the value BU_n is finite, the functions $\overline{U}(\tau)$, for which a solution of (1)–(2) has already been found (in papers ([1],[2],[4])), assume the value $\overline{U}(1/2)$ only at $\tau = 1/2$, and in a neighborhood of $\tau = 0$ the functions $\overline{U}(\tau)$ and $BU_n(\tau)$ have respectively the form $\overline{U}(\tau) = a/\sqrt{\tau} + ...$, $BU_n(\tau) = b/\sqrt{\tau} + ...$, then, for sufficiently small ε, the value 1 is attained by the function $U_{n+1}(\tau)$ only at $\tau = 1/2$.

From (10) it is seen that the operator BU is bounded ($\|BU\| \le B$); from (14) it follows that the constants λ_n are bounded ($\lambda_n < \overline{\lambda}$), if $\varepsilon B/\overline{U}(1/2) < 1$. Using also (13), (14), (12), and denoting

$$\nu = \max\{2a\varepsilon\overline{\lambda},\ 2\varepsilon B,\ a\varepsilon\overline{\lambda}/\overline{U}(1/2),\ \varepsilon B/\overline{U}(1/2)\}, \tag{15}$$

we obtain the estimates

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$$\|U_{n+1} - U_n\| \leq \nu\{\|U_n - U_{n-1}\| + |\lambda_n - \lambda_{n-1}|\},$$

$$|\lambda_{n+1} - \lambda| \leq \nu\{\|U_n - U_{n-1}\| + |\lambda_n - \lambda_{n-1}|\},$$

whence the inequalities follow

$$\|U_{n+1} - U_n\| \leq \xi(2\nu)^n M, \qquad |\lambda_{n+1} - \lambda| \leq \xi(2\nu)^n M,$$

$$\|U_{n+k} - U_n\| \leq \xi(2\nu)^n M/(1 - 2\nu),$$

$$|\lambda_{n+k} - \lambda_n| \leq \xi(2\nu)^n M/(1 - 2\nu),$$

$$\left(\xi = \frac{1}{4\nu}, \quad M = \|U_2 - U_1\| + |\lambda_2 - \lambda_1|\right).$$

Thus, if $2\nu < 1$, i.e., according to (15), for sufficiently small ε, the sequences U_n and λ_n are fundamental and, by virtue of the completeness of the space C and of the space of numerical sequences, each of these sequences converges to a limit $U_*(\tau)$ and λ_*, respectively; moreover, the function $U_*(\tau)$ is continuous.

Let us show that the function $U_*(\tau)$ and the constant λ_* satisfy equation (11). Using (11), (13), and (10), we have

$$\|U_* - \lambda_*\overline{U} + \varepsilon\lambda_* BU_*\| \leq \|U_* - U_{n+1}\| + \|\overline{U}\|\,|\lambda_{n+1} - \lambda_*| +$$

$$+a\varepsilon\lambda_*\|U_* - U_n\| + \varepsilon B|\lambda_* - \lambda_n|. \tag{16}$$

Since every term on the right-hand side of (16) tends to zero as n increases, $\|U_* - \lambda_*\overline{U} + \varepsilon\lambda_* BU_*\| = 0$, whence it follows that U_* and λ_* satisfy (11).

We shall prove that $\|U_*\| = 1$. By means of the relations $\|U_n\| = 1$ and $U_n(1/2) = 1$, we find

$$\|U_*\| \leq \|U_n\| + \|U_* - U_n\| \leq 1, \qquad U_*(1/2) = U_n(1/2) + U_*(1/2) - U_n(1/2) = 1 + U_*(1/2) - U_n(1/2),$$

hence $U_*(1/2) = 1$, $\|U_*\| = 1$. It is easy to show that if $\tau \neq 0$, then $U_*(\tau) > -1$, and if $\tau \neq 1/2$, then $U_*(\tau) < 1$.

We shall prove uniqueness of the solution U_*, λ_* of equation (11). Suppose equation (11) has two solutions U_*, λ_* and U_{**}, λ_{**}. Then from (11) it follows that

$$\|U_* - U_{**}\| \leq \alpha\varepsilon\lambda_*\|U_* - U_{**}\| + \left(\|\overline{U}\| + \varepsilon\|BU_{**}\|\right)|\lambda_* - \lambda_{**}|, \tag{17}$$

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Fig. 1

Figure 1: Fig. 1

$$|\lambda_* - \lambda_{**}| \le \alpha\varepsilon\lambda_*\lambda_{**}\|U_* - U_{**}\|. \tag{18}$$

Substituting (18) into (17), we find the inequality

$$\|U_* - U_{**}\| \le N\|U_* - U_{**}\|(N = \alpha\varepsilon\lambda_*[1 + \lambda_{**}(\|\overline{U}\| + \varepsilon\|BU_{**}\|)]). \tag{19}$$

It is clear that, for sufficiently small ε, $\|U_* - U_{**}\| = 0$, i.e. $U_* = U_{**}$, and then from (11) $\lambda_* = \lambda_{**}$.

Fig. 1

The solution of equation (1) with condition (2) in explicit form for rectangular oscillations $\chi_i(c) = \chi_i = \text{const}$ was found in papers [1,4], and for the case $\chi_i(c) = \chi_i + hc$ $(h > 0)$, in paper [4].

In paper [2] the inverse problem was solved: the functions $\Psi_i[u_i(0,t)]$ are regarded as known as functions of t:

$$\Psi_1[u_1(0,t)] = \psi(t), \qquad \Psi_2[u_2(0,t)] = \varphi(t),$$

and the case in which $\psi(t)$ is defined by the formula

$$\psi(t) = e' + d'(-p/2 + 1 - \tau)^q + f'(\tau - p/2)^{1/2} \tag{20}$$

$$(0 < p < 1,\ e' < 0,\ d' < 0,\ f' > 0,\ 1/2 < q < 1).$$

The function $\varphi(t)$ has an analogous form; moreover, certain conditions are imposed on the constants entering the expressions for $\psi(t)$ and $\varphi(t)$ [2].

Thus, in the present article, for the symmetric case, it has been shown how to find the solution of equation (1) with condition (2), where the function $\chi(c)$ differs little from that occurring in these three problems.

Let us dwell in more detail on the case when the prescribed function $F(U)$ in a neighborhood of $U = 1$ has the form $F(U) = F(1) + B(1-U)^q + \cdots$, and the function $\overline{U}(\tau)$ represents the solution of the inverse problem $\mu F_0[U(\tau)] = \psi(t)$, where $\psi(t)$ is given by formula (20), and, by symmetry, $p = 1/2$ [2]. Near the value $\tau = 1/2$,

$$\overline{U} = 1 + \overline{C}(\tau - 1/2) + \cdots, \qquad \overline{C} = \sqrt{DT}\,[a_1 e' + a_2(q)d' + a_3 f']/\mu \quad {}^2.$$

Using the notation adopted by us, we find, as shown in [2], that

$$\overline{C} = (d'/\mu B)^{1/q}.$$

Taking into account that $F(1) = F_0(1)$, $F(-1) = F_0(-1)$, it is easy to show that, from the prescribed values B, $F(1)$, and $F(-1)$, one can uniquely find the

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constants e'/μ, d'/μ, and f'/μ. Putting $\varepsilon = 0$ in (11), from (14) we find λ, i.e. the period T as a function of μ and u_+, and from (11) the solution of the problem under consideration in the first approximation, when $F(U) = F_0(U)$ (the dotted curve in Fig. 1). Now the problem is reduced to determining the solution U of equation (11), where $\varepsilon\zeta(U) = F(U) - F_0(U)$ is a known function, with $\zeta'(1) = 0$, so that condition (9) is satisfied.

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[1] A. Ya. Gokhshtein, DAN, **140**, No. 5 (1961).
[2] N. N. Kochina, PMM, **28**, issue 4 (1964).
[3] G. M. Fikhtengol' ts, *Course of Differential and Integral Calculus*, 2, 1948.
[4] N. N. Kochina, DAN, **165**, No. 5 (1965).

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