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1967

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Abstract

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ANALYTIC CAPACITY AND THE CAUCHY INTEGRAL

(Presented by Academician A. N. Kolmogorov, 14 IX 1966)

The paper gives an estimate of the Cauchy integral over a sufficiently smooth contour in terms of the analytic capacity of the set of singular points of the function enclosed by this contour. This estimate proves useful in the theory of approximation of functions by rational fractions.

§ 1. **Formulation of the results and some consequences.**

Definition. A rectilinear polygonal curve γ, having no self-intersections and consisting of a finite number of adjoining links (Jordan arcs $\gamma_1, \dots, \gamma_n$), is called a **Lyapunov curve** if each link γ_i has a tangent at every point and the argument $\arg\tau(z,\gamma_i)$ of the tangent vector $\tau(z,\gamma_i)$ of the arc γ_i at the point z, as a function of the point $z \in \gamma_i$, satisfies a Hölder condition with constant L and exponent α, i.e., if for all z_1 and z_2 from γ_i,

$$|\arg\tau(z_1,\gamma_i) - \arg\tau(z_2,\gamma_i)| \le L|z_1 - z_2|^{\alpha}.$$

A domain whose boundary consists of a finite number of Lyapunov curves will be called a **Lyapunov domain**.

Notation. $A(e,m)$ is the set of all functions each of which is analytic outside some closed subset of the set e, bounded in modulus by the constant m, and equal to zero at the point at infinity; $C(e,m)$ is the subset of functions continuous on the whole plane from $A(e,m)$; $\gamma(e,f) = \lim_{z\to\infty} zf(z)$ is the residue of the function $f(z)$; $\gamma(e) = \sup|\gamma(e,f)|,\ f \in A(e,1)$, is the analytic capacity of the set e; $\alpha(e) = \sup|\gamma(e,f)|,\ f \in C(e,1)$, is the analytic C-capacity of the set e; ∂g is the boundary of the domain g; $C(g,e,m)$ is the set of functions continuous on $\bar{g} = g \cup \partial g$, bounded on $\bar{g}$ by the constant m, and analytic in $g \setminus e$; $h_1(e)$ is the Hausdorff length of the set e; $\chi(e) = \sup h_1(e')/\gamma(e')$, where the supremum is taken over all closed $e' \subseteq \partial e$ for which $\gamma(e') > 0$.

Theorem 1. *Let g be a Lyapunov domain; e a closed subset of $\bar{g}$, $\bar{g} = g \cup \partial g$; $f(z)$ a function continuous on $\bar{g}$, with modulus of continuity $\omega(\delta)$, analytic in $g \setminus e$. Then*

$$\left|\int_{\partial g} f(z)\,dz\right| \le C(g)\,\omega[\alpha(e)]\,\alpha(e),$$

where $C(g)$ is a constant depending only on the domain g.

From the proof of Theorem 1 it is evident that $C(g)$ depends only on n (the total number of boundary links), on L and α (the Hölder constant and exponent for the links), and on $\chi(g)$. It seems plausible that $C(g)$ is completely determined by the number $\chi(g)$. As $h_1(\partial g)$ increases, the quantity $C(g)$ becomes arbitrarily large (see (4)). In [4], for every k, a domain g_k bounded by a smooth curve, a set $e \subset g_k$, and a function

$$f(z) \in C(g, e, 1)$$

are constructed such that

$$\left| \int_{+\partial g_k} f(z)\, dz \right| \geq k\gamma(e).$$

Relying on this example, ···

for example, one can construct a domain with a rectifiable boundary for which Theorem 1 is no longer true.

Theorem 2. Let g be a Lyapunov domain; let e be a closed subset of g; let $f(z)$ be a function bounded on $\bar{g}$ by the constant m, analytic in $g \setminus e$, and continuous almost everywhere on ∂g. Then

$$\left| \int_{\partial g} f(z)\, dz \right| \leq C(g) m \gamma(e).$$

This theorem follows easily from Theorem 1. For the case of a domain bounded by an analytic curve, Theorem 2 was proved by M. S. Melnikov (see ([1])); however, for some of the simplest types of domains, for example for a square, this estimate of the integral had not been proved. If the boundary of the domain g consists of segments, then $C(g) \leq C_0 n$, where C_0 is an absolute constant.

For a simply connected domain bounded by a smooth Lyapunov curve (a curve having no points of discontinuity of the tangent argument), the proof of Theorem 1 by means of a conformal mapping of the domain onto a disk reduces to the estimate, already obtained by M. S. Melnikov, of the Cauchy integral over a circle. In doing so, it is only necessary to trace how the analytic capacity of a set changes under the corresponding conformal mapping.

Difficulties appeared only in considering domains whose boundaries contain points of discontinuity of the tangent argument. In this case one cannot use conformal mappings, since under the mapping of such domains onto a disk the capacity of sets concentrated near the corner points of the boundary increases inadmissibly strongly.

We formulate two corollaries of Theorem 1.

Theorem 3. Let e be a closed bounded set; let $f(z)$ be a function continuous in the whole plane with modulus of continuity $\omega(\delta)$. In order that $f(z)$ be uniformly approximable on e, with arbitrary accuracy, by rational functions, it is necessary and sufficient that for every square $\xi(z,\delta)$ with center at the point z and side δ the inequality

$$\left|\int_{\partial\xi(z,\delta)} f(z)\,dz\right| \leq C\omega(\delta)\gamma[\xi(z,\delta)\setminus e],$$

hold, where C is a constant common to all z and δ.

The sufficiency of the formulated conditions was proved in (^{2}); the necessity follows easily from Theorem 2.

Theorem 4. If the inner boundary of a closed bounded set e lies on a countable number of Lyapunov curves, then every function continuous on e and analytic at the interior points of the set e can be uniformly approximated on e, with arbitrary accuracy, by rational fractions.

This follows from Theorem 1 and Theorems 3 and 7 of (^{3}).

§ 2. Change of capacity under conformal deformations of a set.

Lemma 1. Let g be a Lyapunov domain; let $z' = \varphi(z)$ be a conformal mapping of the domain g onto the domain $g' = \varphi(g)$, continuously extendable to $\bar{g}$ and one-to-one on $\bar{g}$, and such that $d\varphi/dz$ satisfies on $\bar{g}$ a Hölder condition with exponent $\alpha_1 > 0$ and constant L_1. Then for every closed set $e \subset \bar{g}$ and $e' = \varphi(e)$ the inequality $\alpha(e') \leq M(g,\varphi)\alpha(e)$ holds, where $M(g,\varphi)$ is a constant independent of e.

Lemma 2. Let g be a simply connected Lyapunov domain bounded by a smooth curve (a curve having no points of discontinuity of the tangent argument). Then for every $e \subset \bar{g}$ and $f(z) \in C(e,m)$ the inequality

$$\left|\int_{\partial g} f(z)\,dz\right| \leq C_1(g)m\alpha(e),$$

holds, where $C_1(g)$ is a constant depending only on g.

Proof. By Kellogg' s theorem there exist conformal mappings $z' = \varphi(z)$ of the closed domain g onto the closed disk σ, whose derivative satisfies a Hölder condition. Therefore, from Lemma 1 we obtain that for every $e \subset g$ and $e' = \varphi(e)$ the inequality $\alpha(e') \leq M(g,\varphi)\alpha(e)$ holds. The function $1/(d\varphi/dz)$ is analytic in g and bounded. Since, for every closed set lying strictly inside the disk, the estimate of the integral has been proved, i.e., for every $\psi(z) \in C(\sigma, e', m)$, e' contains no points of the circle,

$$\left|\int_{\partial\sigma} \psi(z')\,dz'\right| \le C(g)m\alpha(e') \quad \text{(see (1))},$$

we have

$$\left|\int_{\partial g} f(z)\,dz\right| = \left|\int_{\partial g} \frac{f(z)}{d\varphi/dz}\frac{d\gamma}{dz}\,dz\right| =$$

$$= \left|\int_{\partial\sigma} \frac{f[\varphi^{-1}(z')]\,dz'}{\dfrac{d}{dz'}\varphi[\varphi^{-1}(z')]}\right| \le C(g)\max_{z'\in\sigma}\left|\frac{f[\varphi^{-1}(z')]}{\dfrac{d}{dz'}\varphi[\varphi^{-1}(z')]}\right|\alpha(e') \le$$

$$\le C_5(g)m\alpha(e') \le C_5(g)M(g,\varphi)m\alpha(e) = C_1(g)m\alpha(e).$$

Thus, if the closed set e lies strictly inside the domain g and $f(z) \in C(g,e,m)$, then

$$\left|\int_{\partial g} f(z)\,dz\right| \le C_1(g)m\alpha(e).$$

If e is an arbitrary closed set in $\bar{g}$ and $f(z) \in C(g,e,m)$, then, by Theorem 7 of (3), $f(z)$ can be uniformly approximated with arbitrary accuracy by a function $f^*(z) \in C(g,e,m)$, analytic in a neighborhood of the boundary of the domain g. Since for $f^*(z)$ the estimate of the integral has already been obtained, i.e.,

$$\left|\int_{\partial g} f^*(z)\,dz\right| \le C_1(g)(m+\varepsilon)\alpha(e),$$

it follows also for the function $f(z)$ that

$$\left|\int_{\partial g} f(z)\,dz\right| \le C_1(g)m\alpha(e).$$

The lemma is proved.

§ 3. **Estimate of the Cauchy integral over the boundary of a square.**

We shall now prove Theorem 1 for the case when the domain g is a square.

If $\alpha(e)$ is commensurable with the side of the square, then the corresponding integral is easily estimated in the required manner by the length of the boundary of the square. Therefore, in what follows we shall assume that $\alpha(e)$ is small in comparison with the side of the square. We shall denote by $\lambda_1, \lambda_2, \ldots$ absolute constants. Let $f(z) \in C(g, e, m)$ be continuous in the whole plane and have modulus of continuity $\omega(\delta)$. Represent the function $f(z)$ in the form $f(z) = \sum_k f_k^\alpha(z)$, where

$$f_k^\alpha(z) = \frac{1}{2\pi i} \int \frac{\partial f}{\partial \bar{\xi}} \frac{g_{k,n}(\xi)}{\xi - z} \, d\bar{\xi} \, d\xi; \quad n = \left[\frac{1}{\alpha(e)}\right]$$

is a natural number; $\{g_{k,n}(\xi)\}$ is the special partition of unity constructed in paper (3).

Denote by v_k the closed square which is the closure of the support of the function $g_{k,n}(\xi)$. From the construction of the function $g_{k,n}(\xi)$ it follows that every point of the plane (in particular, every vertex of the square g) belongs simultaneously to no more than λ_1 squares from $\{v_k\}$, and $f_k^\alpha(z) \in C(e'_k, \lambda_2\omega(\alpha))$, where $\alpha = \alpha(e)$; $e'_k = (e \cap v_k) \cup (v_k \setminus g)$; since $e_k = e \cap v_k \subset v_k$ and the squares $\{v_k\}$ cover the plane with multiplicity not exceeding λ_1, it follows, by Lemma 3 of paper (3), that

$$\sum_k \alpha(e_k) \leq \lambda_3 \alpha(e).$$

Since $f(z) = \sum_k f_k^\alpha(z)$, we have

$$\left| \int_{\partial g} f(z) \, dz \right| \leq \sum_k \left| \int_{\partial g} f_k^\alpha(z) \, dz \right|.$$

1. If $v_k \setminus g = \emptyset$, then $f_k^\alpha(z) \in C(e_k, \lambda_2\omega(\alpha))$, and therefore

$$\left| \int_{\partial g} f_k^\alpha(z) \, dz \right| = |2\pi i \gamma\,(e_k, f_k^\alpha)| \leq 2\pi \max_z |f_k^\alpha(z)| \, \alpha(e_k) \leq \lambda_4 \omega(\alpha) \alpha(e_k).$$

2. Let s_j be one of the four sides of the square g, and let the number k be such that v_k contains none of the vertices of the square g and $\partial g \cap v_k = s_j \cap v_k$. Let d_j be a smooth Lyapunov curve containing s_j as its subset and enclosing g; let g_j be the domain bounded by this curve. Since in the case under consideration $f_k^\alpha(z) \in C(g_j, e_k, \lambda_2\omega(\alpha))$, it follows, by Lemma 2, that

$$\left| \int_{\partial g} f_k^\alpha(z) \, dz \right| = \left| \int_{\partial g_j} f_k^\alpha(z) \, dz \right| \leq C_1(g_j) \lambda_2 \omega(\alpha) \alpha(e_k) = \lambda_5 \omega(\alpha) \alpha(e_k).$$

3. The number of values of the index k that do not satisfy the conditions of items 1, 2 does not exceed a certain absolute constant λ_6. For each of these values

$$\left|\int_{\partial g} f_k^\alpha(z)\,dz\right| \leq \max_z |f_k^\alpha(z)|\{h_1(\partial v_k)+h_1(v_k\cap\partial g)\} \leq \lambda_2\omega(\alpha)\lambda_7\alpha(e) = \lambda_8\omega(\alpha)\alpha(e).$$

From the inequalities of items 1-3 and the inequality $\sum_k \alpha(e_k) \leq \lambda_3\alpha(e)$, we obtain

$$\left|\int_{\partial g} f(z)\,dz\right| \leq \sum_k \left|\int_{\partial g} f_k^\alpha(z)\,dz\right| \leq$$

$$\leq \sum_k \lambda_4\omega(\alpha)\alpha(e_k) + \sum_k \lambda_5\omega(\alpha)\alpha(e_k) + \lambda_6\lambda_8\omega(\alpha)\alpha(e) \leq \lambda_9\omega(\alpha)\alpha(e),$$

i.e., for the case where g is a square, the theorem is proved.

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Academy of Sciences of the USSR

Received
14 IX 1966

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Note: Figure translations are in progress. See original paper for figures.

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