ON THE STABILITY OF A BOUNDARY-VALUE PROBLEM IN THE THEORY OF ANALYTIC FUNCTIONS

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Abstract

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MATHEMATICS

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ON THE STABILITY OF A BOUNDARY-VALUE PROBLEM IN THE THEORY OF ANALYTIC FUNCTIONS

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1. Consider the boundary-value problem: to find two functions $\varphi^+(z)$ and $\varphi^-(z)$, analytic respectively in the domains S^+ and S^- into which the unit circle Γ divides the plane, representable by Cauchy integrals with boundary values from $\mathcal{L}_p(\Gamma)$, satisfying on Γ the boundary condition

$$\varphi^+(t) = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + c(t), \tag{1}$$

where $a(t)$ is a continuous function on Γ, $|a(t)| > 0$; $b(t)$ is a bounded measurable function on Γ; $c(t) \in \mathcal{L}_p(\Gamma)$, $p > 1$.

The boundary-value problem (1) was first formulated by A. I. Markushevich ([1]) and subsequently attracted attention many times. The first results on the solvability of problem (1) in the class of piecewise meromorphic functions were obtained by N. P. Vekua ([2]). B. V. Boyarskii ([3]), and then L. G. Mikhailov ([4]), constructed the Noether theory of problem (1) and clarified the picture of solvability of problem (1) in the case $|a(t)| > |b(t)|$, called elliptic by L. G. Mikhailov. In addition, L. G. Mikhailov studied the parabolic case $|a(t)| = |b(t)|$, assuming that $a(t), b(t) \in H(\Gamma)$. I. Kh. Sabitov ([5]), using the results of L. G. Mikhailov, obtained some conclusions on the solvability of problem (1) when the conditions of ellipticity or parabolicity are not satisfied.

In the present note a method is proposed for studying problem (1), based on the use of a deep analogy between problem (1) and the corresponding Riemann problem for a system of two pairs of unknown functions. This approach makes it possible, under fairly general assumptions on the given functions, to clarify the question of the stability of problem (1) with respect to small variations of the given functions. At the same time, the results of papers ([3,4]) receive a simple explanation, and L. G. Mikhailov' s theorem concerning the parabolic case is substantially sharpened. Along the way, two sufficient criteria are established for the stability of the system of partial indices of the Riemann boundary-value problem for n pairs of functions.

2. Using the method of analytic continuation of N. I. Muskhelishvili ([6]), we reduce problem (1) to the Riemann problem

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \qquad t \in \Gamma, \tag{2}$$

where

$$G(t) = G_1(t)G_2(t), \qquad G_1(t) = \begin{Vmatrix} \overline{a^{-1}(t)} & 0 \\ 0 & a^{-1}(t) \end{Vmatrix},$$

$$G_2(t) = \begin{Vmatrix} |a(t)|^2 - |b(t)|^2 & b(t) \\ -\overline{b(t)} & 1 \end{Vmatrix},$$

$$g(t) = \left\{ [a(t)c(t) - b(t)\overline{c(t)}]\overline{a^{-1}(t)},\ -c(t)a^{-1}(t) \right\}.$$

It follows from ([7]) that problem (1) is solvable if and only if problem (2) is solvable, and the number of solutions of problem (1) (over the field of real coefficients) is equal to the number of solutions of problem (2) (over the field of complex coefficients). To each solution $\varphi(z) = \{\varphi^+(z), \varphi^-(z)\}$, $\varphi^-(\infty) = 0$, of problem (1) there corresponds the solution $\Phi_1(z) = \{\varphi^+(z), \varphi^-(z)\}$, $\Phi_2(z) = \{\overline{\varphi^-(1/z)}, \overline{\varphi^+(1/z)}\}$ of problem (2), which evidently satisfies the conditions $\overline{\Phi_2^+(z)} = \Phi_1^-(1/z)$, $\Phi_1^-(\infty) = 0$.

Conversely, if $\{\Phi_1(z), \Phi_2(z)\}$ is any solution of problem (2) bounded at infinity, then the functions $\varphi^+(z) = \Phi_1^+(z) + \overline{\Phi_2^-(1/\bar{z})}$, $\varphi^-(z) = \Phi_1^-(z) + \overline{\Phi_2^+(1/\bar{z})}$, under the condition $\Phi_1^-(\infty) + \overline{\Phi_2^+(0)} = 0$, give a solution of problem (1) that vanishes at infinity.

Definition. We shall call the boundary-value problem (1) **stable** (**unstable**) if the corresponding Riemann problem (2) has a stable (unstable) system of partial indices.

The matrix $G(t)$ is nonsingular by virtue of the condition $|a(t)| > 0$. Since the total index of problem (2) is equal to $2\varkappa$, where $\varkappa = \operatorname{Ind} a(t)$, it follows, by the theorem established by I. Ts. Gokhberg, M. G. Krein ([8]) and B. V. Boyarskii ([9]), that the only stable system of partial indices of problem (2) is the system $(\varkappa, \varkappa)$. Let l and p denote, respectively, the numbers of linearly independent solutions and solvability conditions of problem (1).

3. **Theorem 1.** *If $|a(t)| > |b(t)|$, then the boundary-value problem (1) is stable.*

Proof. It suffices to establish that the matrix $G(t)$ admits the representation (factorization)

$$G(t) = X^+(t)\Lambda(t)X^-(t), \tag{3}$$

where

$$\Lambda(t) = \begin{Vmatrix} t^{\varkappa} & 0 \\ 0 & t^{\varkappa} \end{Vmatrix},$$

and $X^{+}(t)$ and $X^{-}(t)$ are the boundary values of the matrices $X^{+}(z)$ and $X^{-}(z)$, analytic respectively in the domains S^{+} and S^{-} and not vanishing at any point of the plane, including $z = \infty$. We have $G_1(t) = X_1^{+}(t)\Lambda(t)X_1^{-}(t)$. Denote $\nu = \min(1, m)$, $m = \inf_{\Gamma}\{|a(t)|^2 - |b(t)|^2\}$.

Since $|a(t)| > |b(t)|$ on Γ, we have $\nu > 0$, and

$$\operatorname{Re} G_2(t) = \frac{1}{2}\left[G_2(t) + \overline{G_2'(t)}\right] \geq \|\nu\delta_{ik}\|,$$

where δ_{ik} is the Kronecker symbol. Consequently ([10]), $G_2(t) = X_2^{+}(t)X_2^{-}(t)$. Since $X_3^{+}(t)$ commutes with $X_1^{-}(t)$ and $\Lambda(t)$, we obtain (3), where $X^{\pm}(z) = X_1^{\pm}(z)X_2^{\pm}(z)$. The theorem is proved.

Thus the ellipticity condition of B. V. Boyarskii–L. G. Mikhailov is a sufficient condition for the stability of problem (1).

4. **Theorem 2.** *If* $|a(t)| = |b(t)| > 0$ *and* $k = \operatorname{Ind} b(t) \geq 0$*, then the boundary-value problem (1) is stable, with* $l = \max(0, 2\varkappa)$*,* $p = \max(0, -2\varkappa)$*.*

Proof. Consider the Riemann boundary-value problem with matrix $G_2(t)$

$$\tilde{\Phi}_1^{+}(t) = b(t)\tilde{\Phi}_2^{-}(t), \tag{4_1}$$

$$\tilde{\Phi}_2^{+}(t) = -\overline{b(t)}\,\tilde{\Phi}_1^{-}(t) + \tilde{\Phi}_2^{-}(t). \tag{4_2}$$

The general solution of problem (4_1) has the form ([11])

$$\tilde{\Phi}_1^{+}(z) = b^{+}(z)P_{k-1}(z), \qquad \tilde{\Phi}_2^{-}(z) = z^{-k}b^{-}(z)P_{k-1}(z), \qquad P_{k-1}(z) = \sum_{l=0}^{k-1} c_l z^l,$$

where $b^{+}(t)[b^{-}(t)]^{-1} = t^{-k}b(t)$; $b^{+}(z)$ and $b^{-}(z)$ are functions analytic and nonzero in S^{+} and S^{-}. The solvability conditions for problem (4_2) can be written in the form of a system of linear algebraic equations

$$\sum_{l=0}^{k-1} c_l \int_{|t|=1} |b^{-}(t)|^2\, t\bar{t}^{\,k-s-1}\, d\varphi = 0, \qquad s = 0, 1, \ldots, k-1,$$

whose determinant is nonzero, since it is the Gram determinant for a linearly independent system of functions with nonzero weight $|b^{-}(t)|^2$. Consequently,

$c_0 = c_1 = \dots = c_{k-1} = 0$ and $G_2(t) = X_2^+(t)X_2^-(t)$, while for $G(t)$ we again obtain (3). The assertions about the numbers l and p are obvious. The theorem is proved.

Theorem 2 strengthens the corresponding result of L. G. Mikhailov for the case $|a(t)| = |b(t)| > 0$, $\lambda = \operatorname{Ind} a(t) + \operatorname{Ind} b(t) \geq 0$, $\mu = \operatorname{Ind} a(t) - \operatorname{Ind} b(t)$. For the numbers l and p, L. G. Mikhailov obtained in this case the estimates: for $\varkappa \geq 0$, $2\varkappa \leq l \leq \lambda - 1$, $p = 0$; for $\varkappa < 0$, $0 \leq l \leq \lambda - 1$,

$$-2\varkappa \leq p \leq |\mu| + 1.$$

We have $\lambda - \mu = 2 \operatorname{Ind} b(t) > 0$, and theorem 2 is applicable, from which it follows that $l = 2\varkappa$, $p = 0$ for $\varkappa > 0$ and $l = 0$, $p = -2\varkappa$ for $\varkappa < 0$; the remaining cases provided for by L. G. Mikhailov' s estimates are not realized.

5. **Theorem 3.** *If $|a(t)| = |b(t)| > 0$ and $\operatorname{Ind} b(t) = -k < 0$, then the boundary-value problem (1) is unstable.*

Proof. Since $\operatorname{Ind} b(t) < 0$, the general solution of problem $(4_1), (4_2)$ depends on k arbitrary constants, while the corresponding homogeneous problem has the same number of solvability conditions. Consequently, the partial indices of problem (2) form the unstable combination $(\varkappa + k, \varkappa - k)$. The theorem is proved.

6. The methods used to prove theorems 1 and 2 make it possible to establish the following sufficient criteria for the stability of the system of partial indices of the Riemann boundary-value problem for n pairs of unknown functions.

Theorem 4. *The Riemann boundary-value problem with matrix $G(t) = A(t)B(t)$, where $A(t) = \|a(t)\delta_{ik}\|$, $B(t) = \|b_{ik}(t)\|$ is a skew-symmetric matrix $(b_{ik}(t) = -b_{ki}(t))$,*

$$\inf_{|t|=1} \operatorname{Re} b_{ii}(t) > 0, \qquad i, k = 1, 2, \dots, n,$$

has the stable system of partial indices

$$\varkappa_1 = \varkappa_2 = \dots = \varkappa_n = \operatorname{Ind} a(t).$$

Theorem 5. *The Riemann boundary-value problem with matrix $G(t) = A(t)B(t)$, where $A(t) = \|a(t)\delta_{ik}\|$, $B(t) = \|b_{ik}(t)\|$, $b_{ik}(t) = 0$ for $i < k$, $\operatorname{Im} b_{ik}(t) = 0$ for $i > k$,*

$$b_{2s,2s}(t) = \overline{b_{2s-1,2s-1}(t)} \neq 0, \qquad \operatorname{Ind} b_{2s-1,2s-1}(t) \geq 0, \qquad s = 1, 2, \dots, n/2,$$

$$i, k = 1, 2, \dots, n,$$

has the stable system of partial indices

$$\varkappa_1 = \varkappa_2 = \dots = \varkappa_n = \operatorname{Ind} a(t).$$

Let us note that theorems 1-3 can be extended to the more general problem

$$a(t)\varphi^+(t) + b(t)\overline{\varphi^+(t)} = c(t)\varphi^-(t) + d(t)\overline{\varphi^-(t)} + g(t)$$

and to the corresponding singular integral equation.

The method proposed here is convenient for the effective solution of problem (1) (for example, in the case when the prescribed functions are rational), since it makes it possible to use for this purpose the corresponding algorithms developed for the Riemann problem. The method applied here also opens a way to obtaining more detailed information, than that given by Noether' s theorems, about the solvability of problem (1) for a system of n pairs of functions. Incidentally, the stability conditions here are considerably more complicated than in the case of a single unknown function.

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Note: Figure translations are in progress. See original paper for figures.