RATIONAL G-SURFACES

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Abstract

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1. In this note we set forth results on the Cremona group which supplement and strengthen a number of theorems from the author' s paper ([1]). The method of adjunction, going back to Castelnuovo and Enriques, is suitable for studying both rational surfaces over a nonclosed field and finite abelian subgroups of the Cremona group, and most of the results are formulated and proved in the same way. This leads to the notion of a G-surface, which we describe below. Classification Theorem 1 refines and extends to the case of G-surfaces the corresponding results of ([1]). Theorem 2 contains information about the representation of the group G in the Picard group of the corresponding surface. Theorems 3, 4, and 5 give a complete birational classification of del Pezzo surfaces of smaller degrees, and also describe the connection between rational points on them and their birational automorphisms. Finally, Theorems 6, 7, and 8 reproduce and partially explain the classical constructions showing the unirationality of del Pezzo surfaces of smaller degrees.

2. By a G-variety we shall mean a composite object consisting of an algebraic variety V over a field k and a group G acting on $V \otimes \bar{k}$. Here V, G, k must satisfy the conditions of one of the following two types.

a) **The algebraic case.** The field k is assumed to be perfect, G is the Galois group of the algebraic closure $\bar{k}/k$; G acts on $V \otimes \bar{k}$ through the second factor.

b) **The geometric case.** The field k is algebraically closed; G is a finite group; the structure of a G-variety on V is given by a homomorphism $G \to \mathrm{Aut}_k(V) = \mathrm{Aut}_k(V \otimes \bar{k})$.

Let two G-varieties of the same type, V_1, V_2, be given. By a G-morphism $f : V_1 \to V_2$ of these varieties we mean any of their k-morphisms in the algebraic case and any k-morphism compatible with the action of the group G in the geometric case. A rational and a birational G-map of two G-varieties of the same type are defined analogously. A G-variety V is called rational if it is nonsingular and regular, and $V \otimes \bar{k}$ is birationally equivalent to projective space over $\bar{k}$. The principal object of study in this note is rational G-surfaces; we are interested mainly in those of their properties which are invariant under birational G-maps.

In the algebraic case these are birational invariants over the ground field; in the geometric case they are properties of the representation $G \to \mathrm{Cr}$ that do not change under conjugation inside the Cremona group Cr.

3. We shall describe a certain class of rational G-surfaces, the elements of which we shall call standard models.

a) **G-surfaces with a rational pencil.** A G-surface F belongs to this class if there exists a G-curve C of genus zero, nonsingular and regular, and a G-morphism $f : F \to C$, whose general geometric fiber is geometrically reduced and irreducible and has genus zero. In addition, it is required that among the G-orbits of the set of irreducible exceptional-

curves of genus zero lying in the fibers of the morphism $f \otimes \bar{k}$ that are not contractible.

b) **Nonsingular del Pezzo surfaces.** A G-surface F belongs to this class if the anticanonical sheaf Ω_F^{-1} is ample and the group $\operatorname{Pic} F$ is cyclic (this definition differs from that adopted in [1]).

c) **Singular del Pezzo G-surfaces.** A G-surface F belongs to this class if $(\Omega_F \cdot \Omega_F) > 0$, the group $\operatorname{Pic} F$ is generated by Ω_F and by classes of curves X with the properties $p_a(X) \leq 0$, $(X \cdot \Omega_F) = 0$, and among such Ω-invariant curves not one is geometrically connected.

For any del Pezzo surface F, the number $(\Omega_F \cdot \Omega_F)$ is called its degree.

Theorem 1. *Every G-surface in the algebraic case, and every G-surface with abelian group G in the geometric case, is G-birationally equivalent to one of the standard models.*

4. Let F be a standard G-surface. The group G acts on the group

$$N(F) = \operatorname{Pic}(F \otimes \bar{k}).$$

Let $\omega_F \in N(F)$ be the class of the canonical sheaf; it is G-invariant. In the case where F is a surface with a rational pencil, there is one more obviously invariant element $\lambda \in N(F)$: the class of a fiber of the structural G-morphism. Put

$$N_0(F) = \{\mu \in N(F) \mid (\mu \cdot \lambda) = (\mu \cdot \omega_F) = 0\},$$

if F is a surface with a pencil, and

$$N_1(F) = \{\mu \in N(F) \mid (\mu \cdot \omega_F) = 0\},$$

if F is a del Pezzo surface. Consider the representation of the group G induced on the spaces

$$S_0(F) = N_0(F) \otimes \mathbf{R}$$

and

$$S_1(F) = N_1(F) \otimes \mathbf{R},$$

respectively.

Theorem 2. *The intersection index (with the opposite sign) defines on $S_0(F)$ and $S_1(F)$ the structure of a Euclidean space. The image of the group G in the corresponding orthogonal group is contained in a certain finite group Γ, generated by reflections. In Witt' s notation [2], the group Γ has the following type.*

If F is a standard surface with a pencil, then Γ has type C_m^*, where

$$m = 8 - (\Omega_F \cdot \Omega_F).$$

If F is a standard del Pezzo surface of degree $(\Omega_F \cdot \Omega_F) = n$, the type of Γ is indicated in the following table:

n	1	2	3	4	5	6	7
Γ	E_8^*	E_7^*	B_6^*	B_5^*	A_4^*	$A_2^* \times Z_2$	Z_2

(Z_2 is the cyclic group of order two acting on a line; for $n = 8$ and $n = 9$ the group Γ is trivial.)

5. On nonsingular del Pezzo surfaces of degree 1 and 2 there are canonical G-involutions s, t, inducing on $S_1(F)$ a symmetry with respect to the origin. Let F be any nonsingular del Pezzo surface of degree i. A finite set of closed points M on $F \otimes \bar{k}$ will be called nonspecial if, after the monoidal transformation with center in M, $F \otimes \bar{k}$ is again transformed into a nonsingular del Pezzo surface F'. If F' has degree 1 or 2, then the canonical involution s or t on F' induces a certain birational automorphism on $F \otimes \bar{k}$, which we shall denote by the symbol s_M or t_M, respectively. If M is G-invariant, then s_M and t_M are birational G-automorphisms.

Theorem 3. *Let F be a standard nonsingular del Pezzo G-surface of degree 1, 2, or 3. Then it is not birationally G-equivalent to any standard G-surface with a rational pencil.*

Theorem 4. *Let F be as in Theorem 3, and let F' be any standard del Pezzo G-surface (a priori, possibly singular) for which*

$$(\Omega_{F'} \cdot \Omega_{F'}) \geq (\Omega_F \cdot \Omega_F).$$

For every birational G-mapping

$f : F' \to F$ there exists a birational G-morphism $g : F \to F^*$, such that the morphism $g \circ f : F' \to F$ is an isomorphism.

In particular, the birational classification of cubic minimal k-surfaces coincides with their projective classification over k. This makes it possible to answer a question of I. R. Shafarevich ([3]): the cubic surfaces

$$x^3 + y^3 + z^3 = a_i \quad (i = 1, 2)$$

are birationally equivalent if and only if $a_1 a_2^{-1} \in (k^*)^3$.

Theorem 5. *The groups of birational G-automorphisms of standard G-surfaces that are del Pezzo surfaces of degree $n = 1, 2$, or 3 have the following systems of generators. For $n = 1$ all birational G-automorphisms are G-automorphisms; in particular, their group is finite. For $n = 2$ the group is generated by G-automorphisms and automorphisms of the form s_x, where x runs through the nonspecial G-invariant points of $F \otimes \bar{k}$. Finally, for $n = 3$ the group is generated by G-automorphisms and automorphisms of the form $s_{x,y}$ and t_z, where (x, y) (respectively z) runs through the G-invariant nonspecial pairs of points on $F \otimes \bar{k}$ (respectively the G-invariant nonspecial points).*

Remark 1. From Theorem 5 and one result of Segre it follows that the existence of a k-point on a minimal nonsingular cubic k-surface is equivalent to the infiniteness of the group of birational automorphisms of this surface.

Remark 2. Much more precise information can be obtained about the groups of birational automorphisms described in Theorem 5. For example, for $n = 2$ the group generated by the automorphisms s_x is the free product of cyclic groups of order two (s_x) over all x.

6. In this section we restrict ourselves to consideration of the algebraically closed case.

As was shown in ([1]), the existence of a k-point on standard nondegenerate del Pezzo surfaces of degree $n \geq 5$ is equivalent to birational triviality. This is not true for $n \leq 4$; however, the following weaker result holds.

Theorem 6. *Let F be a standard nonsingular k-surface that is a del Pezzo surface of degree $2 \leq n \leq 4$. If there exists a nonspecial k-point x on F, then there exists a rational k-morphism of finite degree $f : \mathbf{P}^2 \to F$.*

Remark. B. Segre showed that for $n = 3$ the assumption that the k-point x is nonspecial is superfluous. I do not know whether this is true for $n = 2$ and 4. In addition, it is unknown whether the theorem is true for $n = 1$.

The classical constructions of f for $n = 4, 3$ lead to morphisms of degree 2 and 6, respectively. The following two theorems explain this.

Theorem 7. *Let F be an arbitrary rational k-surface and let there exist a rational k-morphism $f : \mathbf{P}^2 \to F$ of finite degree. Denote by d the least common multiple of the periods of the groups $H^1(K, N(F))$ over all $K \supset k$. Then the degree of f is divisible by d.*

Theorem 8. *Let F be as in Theorem 6. Then there exist surfaces of this type (at least over some fields k) on which there is a nonspecial k-point and for which the number d has the following values. For $n = 4$, $d = 2$; for $n = 3$, $d = 6$; for $n = 2$, $d = 2^\alpha \cdot 3$, where $\alpha \geq 1$ is some integer, and these are the maximal possible values of d.*

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Note: Figure translations are in progress. See original paper for figures.