



Soviet-era science, translated into English

MATHEMATICS

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.32414>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

Yu. G. Reshetnyak

ISOTHERMIC COORDINATES ON SURFACES OF BOUNDED INTEGRAL MEAN CURVATURE

(Presented by Academician A. D. Aleksandrov on 6 VIII 1966)

1. The concept of a surface of bounded integral mean curvature was introduced by I. A. Danilich ^(1,2). The purpose of the present note is to clarify certain questions connected with the problem of the analytic representation of such surfaces.

In what follows R^n , $n \geq 3$, denotes n -dimensional Euclidean space; R^2 is the plane of the complex variable $z = u + iv$; Q_r is the disk $\{z \in R^2 \mid |z| \leq r\}$.

Let P be an arbitrary surface of disk type in R^n (see ^(3,4)); $X : Q_1 \rightarrow R^n$ a parametrization of the surface P . We shall say that the parametrization X is **isothermic** if it satisfies the following conditions:

I_1 . The coordinates of the vector-function X have in Q_1 generalized first derivatives that are square-summable on every disk Q_r , where $0 < r < 1$.

I_2 . For almost all $z \in Q_1$ the equalities hold:

$$[X_u(z)]^2 = [X_v(z)]^2, \quad X_u(z)X_v(z) = 0.$$

A parametrization $X : Q_1 \rightarrow R^n$ of the surface P is called **δ -subharmonic** if it satisfies the following conditions:

S_1 . The parametrization X is isothermic.

S_2 . The Laplace operator ΔX of the vector-function X , understood in the sense of the theory of generalized functions, is a completely additive vector-function of a set on the σ -ring of Borel subsets of the disk Q_1 , and the total variation of this function on the disk Q_1 is

$$|\Delta X|(Q_1) < \infty.$$

We note that for $n = 3$ in the regular case we have

$$\Delta X(E) = \int_E 2H\nu dS,$$

where H is the mean curvature at a point of the surface, ν is the unit normal to the surface, and dS is the area element.

2. Let P be an arbitrary nondegenerate polyhedral surface of disk type in R^n ; $X : Q_1 \rightarrow R^n$ an isothermic parametrization of P . Then there exists a curvilinear triangulation T of the disk Q_1 such that the image of each of the triangles U_j , $j = 1, 2, \dots, m$, composing T , under the mapping X is a plane triangle in R^n , and the edges of the triangulation are analytic (up to the endpoints) simple arcs in the plane. Let l_i , $i = 1, 2, \dots, p$, be all the interior edges of the triangulation T . The vector-function X is continuous in the disk Q_1 and inside each of the domains U_i is analytic and satisfies there the equation $\Delta X = 0$. The generalized function ΔX is therefore concentrated on the set $l_1 \cup l_2 \cup \dots \cup l_p$.

In order to describe the generalized function ΔX precisely, let us define certain quantities associated with the polyhedron P . Consider an edge l_i of the triangulation T . Let U_j and U_k be the triangles of the triangulation T whose common boundary is the edge l_i . Consider the triangles $\Delta_j = X(U_j)$ and $\Delta_k = X(U_k)$ and the segment $\lambda_i = X(l_i)$ on the surface P . Let e_{ij} and e_{ik} be unit vectors lying in the planes Δ_j and Δ_k , respectively, perpen-

pendicular to λ_i and directed from λ_i each toward the side of the corresponding triangle. We put

$$\theta_i = (e_{ij}, \widehat{e_{ik}}), \quad h_i = e_{ij} + e_{ik}.$$

By $s_i(E)$ we denote the set function defined as follows: $s_i(E)$ is equal to the Lebesgue measure of the set $X(E \cap l_i)$ on the line λ_i . We put

$$H(E) = \sum_{i=1}^p \theta_i s_i(E), \quad h(E) = \sum_{i=1}^p h_i s_i(E).$$

The generalized function ΔX for a polyhedron coincides with the vector-valued set function $h(E)$. The isothermal parametrization of a polyhedral surface is, therefore, δ -subharmonic.

The quantity $H(P) = H(Q_1)$ is called the **integral mean curvature** of the polyhedron P .

Let P be a nondegenerate surface of disk type in the space R^n . The surface P is called a $(1, 2)$ **surface of bounded integral mean curvature** if there exists a sequence of polyhedra P_m , $m = 1, 2, \dots$, converging to the surface P as $m \rightarrow \infty$ and such that the sequence $\{H(P_m)\}$ is bounded.

3. Lemma. Let P be an arbitrary nondegenerate surface of disk type in the space R^n ; let P_m , $m = 1, 2, \dots$, be a sequence of surfaces of disk type converging to P as $m \rightarrow \infty$. Let $X_m : Q_1 \rightarrow R^n$, $m = 1, 2, \dots$, be parametrizations of the surfaces P_m satisfying the following conditions:

- A. Each of the parametrizations X_m , $m = 1, 2, \dots$, is isothermal.
- B. The parametrizations X_m are all δ -subharmonic, and the sequence $\{|\Delta X_m|(Q_1)\}$ of total variations of the vector-valued functions $\Delta X_m(E)$ on the disk Q_1 is bounded.
- C. There exists a constant $\delta > 0$ such that, for all m , the point $X_m(0)$ is at a distance not less than δ from the boundary contour of the surface P_m .

Then the sequence of vector-functions $\{X_m\}$ is equicontinuously uniformly continuous in the disk Q_1 . If X_0 is the limit of an arbitrary convergent subsequence $\{X_{m_k}\}$, $m_1 < m_2 < \dots < m_k < \dots$, then X_0 is an isothermal δ -subharmonic parametrization of the surface P .

As one of the consequences of this lemma we note:

Theorem 1. If P is a nondegenerate surface of disk type of bounded integral mean curvature, then P admits an isothermal δ -subharmonic parametrization.

A surface P of disk type will be called **minimal** if it admits an isothermal parametrization $X : Q_1 \rightarrow R^n$ such that $\Delta X \equiv 0$.

A simple closed curve Γ in the space R^n is called an **admissible contour** if there exists at least one surface P of disk type such that the area of this surface is finite and the curve Γ is the boundary contour of P . It is known that over every admissible contour one can span at least one minimal surface. Using the lemma, the admissibility condition for the contour can be removed.

Theorem 2. For every simple closed curve Γ in the space R^n there exists at least one minimal surface whose boundary contour coincides with Γ .

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
23 VI 1966

CITED LITERATURE

- ¹ I. A. Danelich, *Sibirsk. matem. zhurn.*, **4**, No. 3, 519 (1963).
- ² I. A. Danelich, *Sibirsk. matem. zhurn.*, **5**, No. 5, 1035 (1964).
- ³ L. Cesari, *Surface Area*, Princeton, 1956.
- ⁴ A. G. Sigálov, *UMN*, **6**, issue 2, 16 (1951).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.