ON THE DIMENSION OF THE PRODUCT OF ORDERED CONTINUA

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Abstract

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ON THE DIMENSION OF THE PRODUCT OF ORDERED CONTINUA

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Let $X_1, X_2, \dots, X_n$ be ordered continua*, and let

$$P_n = \prod_{i=1}^{n} X_i$$

be their topological product. The main result of this note is the following

Theorem. For bicompacta P_n one necessarily has

$$\dim P_n = \operatorname{ind} P_n = \operatorname{Ind} P_n = n.$$

For the proof of this proposition we shall need the following well-known result (see (2)).

Proposition. A normal space X has $\dim X \geq n$ if there exist n pairs of closed sets (C_1^i, C_2^i), $C_1^i \cap C_2^i = \emptyset$ $(i = 1, \dots, n)$, such that for any partitions** B^i $(i = 1, 2, \dots, n)$ one necessarily has

$$\bigcap_{i=1}^{n} B^i \neq \emptyset.$$

If X is an ordered continuum, then by x_1 and x_2 we denote respectively the minimal and maximal points. We divide the proof of our theorem into several assertions.

Assertion 1. $\dim P_n \geq n$.

We shall prove this assertion by induction on the number of factors. Let $n = 2$, i.e. $P_2 = X \times Y$, where X and Y are ordered continua. Then for any open cover ω of the bicompactum P_2 there exists a monotone ω-mapping onto the square I^2 $(I = [0, 1])$, under which the sets $X \times y_1$, $X \times y_2$ and $Y \times x_1$, $Y \times x_2$ are mapped respectively onto opposite sides of the square. Put $C_1^1 = X \times y_1$, $C_2^1 = X \times y_2$,

$C_1^2 = Y \times x_1$, $C_2^2 = Y \times x_2$. Let B^1 be a partition for the pair (C_1^1, C_2^1), and B^2 a partition for the pair (C_1^2, C_2^2). We shall show that

$$B^1 \cap B^2 \neq \emptyset.$$

Thus we shall prove that $\dim P_2 \geq 2$.

Suppose the contrary; let $B^1 \cap B^2 = \emptyset$. Then take open sets $G_1^1 \supset B^1$ and $G_1^2 \supset B^2$ such that: 1) $[G_1^1] \cap [G_1^2] = \emptyset$, 2) $[G_1^i] \cap (C_1^i \cup C_2^i) = \emptyset$ $(i = 1, 2)$. Next consider open sets G_2^i such that $B^i \subset G_2^i \subset [G_2^i] \subset G_1^i$. By the choice of the sets B^i we have $X \setminus B^i = V_1^i \cup V_2^i$, where V_1^i, V_2^i $(V_1^i \cap V_2^i) = \emptyset$ are open and $C_1^i \subset V_1^i$, while $C_2^i \subset V_2^i$ $(i = 1, 2)$.

Consider the sets

$$U_1 = \Gamma \cap (V_1^1 \cap V_1^2), \quad U_2 = \Gamma \cap (V_1^1 \cap V_2^2),$$

$$U_3 = \Gamma \cap (V_2^1 \cap V_1^2), \quad U_4 = \Gamma \cap (V_2^1 \cap V_2^2),$$

where $\Gamma = X \setminus [G_2^1] \setminus [G_2^2]$. Let us now consider the cover ω of the bicompactum P_2, consisting of the sets

$$\{G_1^1, G_1^2, U_1, U_2, U_3, U_4\},$$

and let f be a monotone ω-mapping (for this cover) of $P_2 = X \times Y$ onto the square I^2, under which $X \times y_1$, $X \times y_2$ and $Y \times x_1$, $Y \times x_2$ go respectively into opposite sides of the square I^2. Then: 1) $f(B^i) \cap f(C_j^i) = \emptyset$ $(i, j = 1, 2)$; 2) $f(C_1^i) \cap f(C_2^i) = \emptyset$ $(i = 1, 2)$; 3) $f(B^i)$ is a partition for the pair $f(C_1^i), f(C_2^i)$. Let $y \in f(B^1) \cap f(B^2)$. Then $f^{-1}(y) \cap B^1 \neq \emptyset$, $f^{-1}(y) \cap B^2 \neq \emptyset$, and since f is an ω-mapping, $f^{-1}(y)$ is necessarily contained both in G_1^1 and in G_1^2, while $G_1^1 \cap G_1^2 = \emptyset$, a contradiction.

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* A continuum is a connected bicompactum.

** A set B is called a partition for the pair (C_1, C_2), $C_1 \cap C_2 = \emptyset$, if $X \setminus B = V^1 \cup V^2$, V^1, V^2 are open and $V^1 \cap V^2 = \emptyset$, $C_1 \subset V^1$, $C_2 \subset V^2$.

We have obtained a contradiction, which proves that $B^1 \cap B^2 \neq \phi$. Thus, assertion 1 for P_2 is proved.

Let us outline the proof for the bicompactum P_n (i.e., for a bicompactum which is the product of n ordered continua). Let

$$P_n = \prod_{i=1}^{n} X_i.$$

Denote the end points of the ordered continuum X_i respectively by x_1^i and x_2^i. Consider the sets

$$C_1^i = \prod_{\substack{k=1 \\ k \neq i}}^{n} X_k \times x_1^i, \qquad C_2^i = \prod_{\substack{k=1 \\ k \neq i}}^{n} X_k \times x_2^i.$$

The sets C_1^i and C_2^i $(i = 1, 2, \dots, n)$ are closed and $C_1^i \cap C_2^i = \phi$. It turns out that if the set B^i is a partition for the pair (C_1^i, C_2^i), then

$$\bigcap_{i=1}^{n} B^i \neq \phi.$$

Suppose the contrary, namely let

$$\bigcap_{i=1}^{n} B^i = \phi.$$

Then we construct such an open cover ω of the bicompactum P_n that no element of this cover intersects all the sets B^i $(i = 1, 2, \dots, n)$, and no element intersects both B^i and C_1^i or C_2^i $(i = 1, 2, \dots, n)$. For the given cover ω we construct such a monotone ω-mapping f of the bicompactum P_n onto the n-dimensional cube I^n that the sets $f(C_1^i)$ and $f(C_2^i)$ are $(n-1)$-dimensional opposite faces in I^n. Since the mapping f is monotone and an ω-mapping, the set $f(B^i)$ is a partition for the corresponding pair of opposite $(n-1)$-dimensional faces of the n-dimensional cube I^n. Consequently,

$$\bigcap_{i=1}^{n} f(B^i) \neq \phi,$$

which contradicts the fact that f is an ω-mapping. Thus,

$$\bigcap_{i=1}^{n} B^i \neq \phi,$$

and assertion 1 is proved.

Assertion 2. $\operatorname{Ind} P_n \leq n$.

Lemma 1. Let X be an ordered continuum, and let U be an arbitrary open subset of it; then

$$\operatorname{ind} \operatorname{Fr} U = \operatorname{Ind} \operatorname{Fr} U = 0.$$

Lemma 2. Let

$$P = \bigcup_{i=1}^{s} X_i,$$

where X_i is an ordered continuum. Then

$$\operatorname{ind} P = \operatorname{Ind} P = 1.$$

Let

$$P_n^i = \prod_{k=1}^{n} X_k^i,$$

where X_k^i is an ordered continuum. By $x_{j_k}^i$ $(k = 1, 2, \dots, n;\ j_k = 1$ or $2)$ we denote the minimal and maximal points.

Definition 1. An l-dimensional face $(l < n)$ of P_n^i will be called

$$F_{l(j_k)}^i = \prod_{m=1}^{l} X_{k_m}^i \times x_{j_1}^i \times \cdots \times x_{j_{k_1-1}}^i \times x_{j_{k_1+1}}^i \times \cdots \times x_{j_n}^i.$$

Definition 2. We shall say that $P_n^{i_1}$ intersects $P_n^{i_2}$ regularly with respect to $X_k^{i_1}$ if

$$X_k^{i_1} \times x_{j_1}^{i_1} \times \cdots \times x_{j_{k-1}}^{i_1} \times x_{j_{k+1}}^{i_1} \times \cdots \times x_{j_n}^{i_1} = X_k^{i_1}(j_i^{i_1})$$

$(j_i = 1$ or $2,\ i = 1, 2, \dots, n;\ i \neq k)$ is equal to an $X_l^{i_2}$-factor for $P_n^{i_2}$.

Definition 3. A collection $\{P_n^i\}$ $(i = 1, 2, \dots, s)$ will be called regular with respect to X_k^1 if, for any P_n^k $(1 \leq k \leq s)$, one can find such a sequence of bicompacta $\{P_n^{i_l}\}$ $(i_l = 1, \dots, m;\ m < k)$ that $P_n^{i_1} = P_n^1$, $P_n^{i_1}$ intersects regularly with $P_n^{i_2}$ with respect to X_k^1, $P_n^{i_2}$ intersects regularly with $P_n^{i_3}$ with respect to $X_k^1(j_1^{i_1})$, and so on; $P_n^{i_{m-1}}$ with $P_n^{i_m} = P_n^k$ with respect to

$$X_k^1(j_1^{i_1})(j_1^{i_2}) \cdots (j_1^{i_{m-2}}).$$

Definition 4. We shall say that $P_n^{i_1}$ **intersects** $P_n^{i_2}$ **along some** l**-dimensional face** $F_{l(j_k)}^{i_1}$, if $P_n^{i_1}$ intersects properly with $P_n^{i_2}$ relative to $X_{k_1}^{i_1}, \dots, X_{k_l}^{i_1}$.

Lemma 3. Suppose we have such a sum of bicompacts $\{P_n^i\}$ $(i = 1, 2, \dots, s)$ that each $P_n^{i_1}$ intersects with $P_n^{i_2}$ along an l-dimensional face $(l < n)$. Then

$$\operatorname{Ind}\left(\bigcup_{i=1}^{s} P_n^i\right) \leq n.$$

Proof. We shall carry out the proof by induction on the number n. For a sum of bicompacts $\{P_1^i\}$, Lemma 3 follows from Lemma 2. Suppose that, for a sum of bicompacts $\{P_{n-1}^i\}$ satisfying the condition of Lemma 3, it has been proved that

$$\operatorname{Ind}\left(\bigcup_{i=1}^{s} P_{n-1}^i\right) \leq n - 1.$$

We shall prove Lemma 3 for a sum of bicompacts $\{P_n^i\}$ satisfying the condition of Lemma 3. Let

$$P = \bigcup_{i=1}^{s} P_n^i.$$

Let $F \subset P$ be an arbitrary closed set and OF an arbitrary neighborhood of it. We shall construct $O_1F \subset OF$, whose boundary would be a sum of bicompacts $\{P_{n-1}^l\}$ satisfying the condition of Lemma 3.

1. Consider $F_1 = F \cap P_n^1$ and $OF_1 = OF \cap P_n^1$. Let $X_1^1, X_2^1, \dots, X_n^1$ be ordered continua such that

   $$P_n^1 = \prod_{k=1}^{n} X_k^1.$$

   Take a partition

   $$\beta_1^1 = \{\alpha_1^1 \times \alpha_2^1 \times \dots \times \alpha_n^1\}$$

   of the bicompact P_n^1, where α_i^1 $(i = 1, 2, \dots, n)$ is a partition of X_i^1.

The complement to the sum of those elements of the partition β_1^1 which do not intersect F_1 is an open set. Denote it by G_1. The boundary of G_1 is a sum of bicompacts $\{P_{n-1}^l\}$ satisfying Lemma 3. Take β_2^1 such that $F_1 \subseteq G_2 \subseteq OF_1$.

$k-1$. Suppose that for $P_n^1, \dots, P_n^{k-1}$ there have already been constructed, respectively, such partitions

$$\beta_k^1,\ \beta_{k-1}^2, \dots, \beta_2^{k-1},$$

that:

I. $F_1 \subset G_1 \subset OF_1, \dots, F_{k-1} \subset G_{k-1} \subset OF_{k-1}$ and

$$\Gamma_{k-1} = \bigcup_{i=1}^{k-1} G_i$$

is an open set in

$$\bigcup_{i=1}^{k-1} P_n^i.$$

The set G_i is the complement to the sum of those elements in

$$\beta_{m+1}^{k-m} \quad (1 \le m \le k-1),$$

which do not intersect F_i in P_n^i.

II. If $P_n^{i_1}$ intersects properly with $P_n^{i_2}$ relative to $X_k^{i_1}$, then the partitions $\beta_{k+1-i_1}^{i_1}$ and $\beta_{k+1-i_2}^{i_2}$ in intersection with $X_{k(j_i)}^{i_1}$ give one and the same partition on $X_{k(j_i)}^{i_1}$.

III. The boundary of G_i $(i = 1, 2, \dots, k-1)$ and Γ_{k-1} is a sum of bicompacts $\{P_{n-1}^l\}$ satisfying the condition of Lemma 3.

k. Take P_n^k. Let

$$F_k = F \cap P_n^k, \qquad OF_k = OF \cap P_n^k, \qquad F_{12\dots k} = F \cap \left(\bigcup_{i=1}^{k} P_n^i\right) = \bigcup_{i=1}^{k} F_i,$$

$$OF_{12\dots k} = OF \cap \left(\bigcup_{i=1}^{k} P_n^i\right) = \bigcup_{i=1}^{k} OF_i.$$

Take on P_n^k such a partition

$$\beta_1^k = \{\alpha_1^k \times \dots \times \alpha_n^k\},$$

that $G_{k(1)} \subseteq OF_k$ (see the definition of G_1 in 1). Let

$$P_n^k = \prod_{i=1}^{n} X_i^k.$$

Take X_l^k. Consider the collection $\{P_n^{j_l}\}$ $(l \le k)$ of bicompacts $(P_n^{j_l} = P_n^k)$, which is proper relative to X_l^k and maximal in the sense of properness. Then on $X_{l(j_i)}^k$ two partitions are defined: $\beta_1^k \cap X_{l(j_i)}^k$ and the one that was constructed on it at the $(k-1)$-st step; take their intersection and denote it by $\alpha_l^{k(1)}$. Then on the bicompacts $P_n^1, \dots, P_n^{k-1}$ we again obtain partitions satisfying I, II, III from item $k-1$, if on each $P_n^{j_l}$ we take, on the corresponding factor, the partition $\alpha_l^{k(1)}$.

The partition $\{\alpha_1^{k(1)} \times \alpha_2^{k(1)} \times \dots \times \alpha_n^{k(1)}\}$ on P_n^k we shall denote by β_2^k, and the partitions on $P_n^1, \dots, P_n^{k-1}$, respectively, by $\beta_{k+1}^1, \dots, \beta_3^{k-1}$. Then the partitions $\beta_{k+1}^1, \dots, \beta_2^k$ satisfy conditions I, II, III of item $k-1$ with $k-1$ replaced by k. For $k = n$ we obtain $F_{12\dots n} = F' = \Gamma_n \subset OF_{12\dots n} = OF$, and the boundary Γ_n

consists of the sum of bicompacts $\{P_{n-1}^l\}$ satisfying the condition of Lemma 3. Thus, Lemma 3 is proved.

Proof of assertion 2. Let $F \subset P_n$ be an arbitrary closed set and let OF be an arbitrary neighborhood of it. Take such a partition $\beta = \{\alpha_1 \times \cdots \times \alpha_n\}$ that $G \subset OF$, where G is the complement of the sum of those elements of the partition β which do not meet F. The boundary of G is the sum of bicompacts $\{P_{n-1}^l\}$ satisfying the condition of Lemma 3, i.e. $\operatorname{Ind} P_n \leq n$.

For any bicompact X, $\dim X \leq \operatorname{Ind} X$ (1). From assertions 1 and 2 it follows that for P_n we have $\dim P_n \geq \operatorname{Ind} P_n$. Consequently, $\dim P_n = \operatorname{ind} P_n = \operatorname{Ind} P_n = n$. Theorem 1 is proved.

Note added in proof. Already after submitting the present article for publication I managed to prove, considerably more simply, a more general result:

Theorem. *Let*

$$P = \prod_{i=1}^{n} X_i,$$

where X_i *is a bicompact* $(i = 1, 2, \dots, n)$ *and* $\operatorname{Ind} X_i = 1$. *Then* $\dim P = \operatorname{ind} P = \operatorname{Ind} P = n$.

An exposition of this result will be published in this same journal.

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Note: Figure translations are in progress. See original paper for figures.