ON THE EQUATION $\Delta U[x(t)] + P[x(t)]U[x(t)] = 0$ IN FUNCTION SPACE

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ON THE EQUATION $\Delta U[x(t)]+P[x(t)]U[x(t)]=0$ IN FUNCTION SPACE

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The Laplacian of a functional $\Delta U[x(t)]$—the continual analogue of the Laplace operator of a function of a finite number of variables—was defined by P. Lévy ([1]). Laplace and Poisson equations with such Laplacians were studied by P. Lévy, E. M. Polishchuk ([2]), and the author ([3,4]). In this note we consider the boundary-value problem of the "generalized Laplace equation"

$$\Delta U[x(t)]+P[x(t)]U[x(t)]=0 \tag{1}$$

for a domain $\Omega\cup\Gamma$ in the space C, defined by the inequality $S[x(t)]\leqslant 1$ ($x(t)\in\Omega$, if $S[x(t)]<1$; $x(t)\in\Gamma$, if $S[x(t)]=1$); $S[x(t)]$ is a twice functionally differentiable functional such that its first and second variations have normal form, and $\Delta S[x(t)]$ is a constant positive nonzero number; C is the space of functions $x(t)$ continuous on $[0,1]$ ($x(0)=0$), with Wiener measure ([5]).

1. Let $P[x(t)]$ be a functional of finite degree ([6]); $P_N[x(t)]$ the partial sum of its expansion in the Fourier-Hermite functionals ([7]); $B_{m_1\dots m_N}$ its Fourier-Hermite coefficients;

$$\Psi_{m_1\dots m_N}[x(t)]=\prod_{i=1}^{N}H_{m_i}\left[\int_0^1\chi_i(t)\,dx(t)\right]$$

are the Fourier-Hermite functionals ($m_i=0,1,2,\dots;\ N=1,2,\dots$); $H_m(u)$ are partially normalized Hermite polynomials ($m=0,1,2,\dots$); $\chi_1(t)\equiv\chi_0^{(0)}(t)$, $\chi_i(t)\equiv\chi_n^{(k)}(t)$ for $i=2^n+k$ ($n=0,1,2,\dots;\ k=1,2,\dots,2^n$), $\chi_0^{(0)}(t)$, $\{\chi_n^{(k)}(t)\}$ are the Haar system of functions. Let

$$\Psi^*_{m_1\dots m_N}[x(t),y_1(\tau),\dots,y_N(\tau)]=$$

$$=\prod_{i=1}^{N}H_{m_i}\left[\int_0^1\chi_i(t)\,dx(t)+2(1-S[x])^{1/2}(\Delta S[x])^{-1/2}\,\tilde{y}_i(\tau)\right],$$

$$\tilde{y}_1(\tau) \equiv y_1(\tau), \quad \tilde{y}_i(\tau) \equiv \tilde{y}_n^{(k)}(\tau), \quad \tilde{y}_n^{(k)}(\tau) = \sqrt{2^n}\left(2z_{(2k-1)/2^{n+1}}(\tau) - z_{(2k-2)/2^{n+1}}(\tau) - z_{2k/2^{n+1}}(\tau)\right),$$

$$z_0(\tau) = 0, \qquad z_{p_q}(\tau) = y_q(\tau), \qquad (p_q = 1,\ 1/2,\ 1/4,\ 3/4, \dots;\ q = 1, 2, \dots, N).$$

Lemma 1. The functional

$$\Phi_{m_1 \dots m_N, N}[x(t)] = \int_{C_N} \Psi^*_{m_1 \dots m_N}[x(t), y_1(1), \dots, y_N(1)] \times$$

$$\times \exp\left\{\frac{1 - S[x]}{\Delta S[x]} \int_0^1 \sum_{\mu_1, \dots, \mu_N = 0}^{N} B_{\mu_1 \dots \mu_N} \Psi^*_{\mu_1 \dots \mu_N}[x(t), y_1(s), \dots, y_N(s)]\, ds\right\} dw_{y_1} \cdots dw_{y_N},$$

where the integral is understood as an N-fold Wiener integral on the product space C, satisfies in Ω the equation

$$\Delta U[x] + P_N[x]U[x] = 0.$$

Indeed, having found the second variation and then computed the Laplacian of the functional $\Phi_{m_1 \dots m_N, N}[x]$, we obtain

$$\Delta\Phi_{(m),N}[x] = \int_{C^N} \Bigg\{ \sum_{i=1}^{N} \Big[2^{n_i+1}\sqrt{m_i(m_i-1)}\, a_i \Psi^*_{(m)-2_i}(1)$$
$$-(1-S)^{-1/2}(\Delta S)^{1/2}\sqrt{2m_i}\,\tilde{y}_i(1)\Psi^*_{(m)-1_i}(1)$$
$$+\sum_{\substack{l=1\\ l\neq i}}^{N}\sqrt{2^{n_i+n_l+2}m_i m_l}\, a_{il}\Psi^*_{(m)-1_i-1_l}(1)\Big]$$
$$+\,2(1-S)(\Delta S)^{-1}\sqrt{2^{n_i+1}m_i}\, b_i \Psi^*_{(m)-1_i}(1)$$
$$\times\left[\int_0^1 \sum_{(\mu)=0}^{N} B_{(\mu)} \sum_{l=1}^{N}\sqrt{2^{n_l+1}\mu_l}\, b_l \Psi^*_{(\mu)-1_l}(s)\,ds\right] - \Psi^*_{(m)}(1)\int_0^1 \sum_{(\mu)=0}^{N} B_{(\mu)}\Psi^*_{(\mu)}(s)\,ds$$
$$+\,(1-S)(\Delta S)^{-1}\Psi^*_{(m)}(1)\int_0^1 \sum_{(\mu)=0}^{N} B_{(\mu)} \sum_{i=1}^{N}\Big[2^{n_i+1}\sqrt{\mu_i(\mu_i-1)}\, a_i\Psi^*_{(\mu)-2_i}(s)$$
$$-(1-S)^{-1/2}(\Delta S)^{1/2}\sqrt{2\mu_i}\,\tilde{y}_i(s)\Psi^*_{(\mu)-1_i}(s)$$
$$+\sum_{\substack{l=1\\ l\neq i}}^{N}\sqrt{2^{n_i+n_l+2}\mu_i\mu_l}\, a_{il}\Psi^*_{(\mu)-1_i-1_l}(s)\Big]\,ds$$
$$+\,(1-S)^2(\Delta S)^{-2}\Psi^*_{(m)}(1)\left[\int_0^1 \sum_{(\mu)=0}^{N} B_{(\mu)} \sum_{i=1}^{N}\sqrt{2^{n_i+1}\mu_i}\, B_i\Psi^*_{(\mu)-1_i}(s)\,ds\right]^2\Bigg\} E_N\, dwy_1 \ldots du \qquad (2)$$

where

$$\Psi^*_{m_1\ldots m_N}(\tau) = \Psi^*_{m_1\ldots m_N}[x(t), y_1(\tau), \ldots, y_N(\tau)], \qquad (m) = (m_1, \ldots, m_N),$$

$$(m) - q_i = (m_1, \ldots, m_{i-1}, m_i - q, m_{i+1}, \ldots, m_N),$$

and a_i, a_{il}, b_i are the sums of the coefficients at the corresponding $[\delta x]^2$,

$$E_N = \exp\left\{\frac{1-S}{\Delta S}\int_0^1 \sum_{(\mu)=0}^{N} B_{(\mu)}\Psi^*_{(\mu)}(s)\,ds\right\}.$$

Computing, by P. Cameron's theorem (8), the Wiener integral of the first partial variation with respect to y_i of the functional

$$(i-S)^{-1/2}(\Delta S)^{1/2}\sqrt{2m_i}\Psi^*_{(m)-1_i}(1)E_N,$$

we have

$$(i-S)^{-1/2}(\Delta S)^{1/2}\int_{C^N}\sum_{i=1}^{N}\sqrt{2m_i}\,\tilde{y}_i(1)\Psi^*_{(m)-1_i}(1)E_N\,dwy_1\ldots dwy_N$$
$$=\int_{C^N}\sum_{i=1}^{N}\Bigg\{2^{n_i+1}\sqrt{m_i(m_i-1)}\,a_i\Psi^*_{(m)-2_i}(1)+\sum_{\substack{l=1\\ l\neq i}}^{N}\sqrt{2^{n_i+n_l+2}m_im_l}\,a_{il}\Psi^*_{(m)-1_i-1_l}(1)$$
$$+(1-S)(\Delta S)^{-1}\sqrt{2^{n_i+1}m_i}\,b_i\Psi^*_{(m)-1_i}(1)\int_0^1 s\sum_{(\mu)=0}^{N}B_{(\mu)}\sum_{l=1}^{N}\sqrt{2^{n_l+1}\mu_l}\,b_l\Psi^*_{(\mu)-1_l}(s)\,ds\Bigg\}$$
$$\times E_N\,dwy_1\ldots dwy_N, \tag{3}$$

and by M. Ouchard' s theorem $(9,10)$—for the partial variational derivative of the functional

$$(1-S)^{1/2}(\Delta S)^{-1/2}\Psi^*_{(m)}(1)\sqrt{2\mu_i}\Psi^*_{(\mu)-1_i}(s)E_N$$

with respect to y_i—we obtain

$$(1-S)^{1/2}(\Delta S)^{-1/2}\int_{C^N}\Psi^*_{(m)}(1)\left[\int_0^1\sum_{(\mu)=0}^{N}B_{(\mu)}\sum_{i=1}^{N}\sqrt{2\mu_i}\,\tilde{y}_i(s)\Psi^*_{(\mu)-1_i}(s)\,ds\right]E_N\,dwy_1\ldots dwy_N$$
$$=\int_{C^N}\Bigg\{-2(1-S)^2(\Delta S)^{-2}\Psi^*_{(m)}(1)\int_0^s\sum_{(\mu)=0}^{N}B_{(\mu)}\sum_{i=1}^{N}\sqrt{2^{n_i+1}\mu_i}\,b_i\Psi^*_{(\mu)-1_i}(s)$$
$$\times\int_1^s\sum_{(\mu)=0}^{N}B_{(\mu)}\sum_{l=1}^{N}\sqrt{2^{n_l+1}\mu_l}\,b_l\Psi^*_{(\mu)-1_l}(v)\,dv\,ds+$$

$$+(1-S)(\Delta S)^{-1}\Psi^*_{(m)}(1)\Bigg\{\int_0^1 s\sum_{(\mu)=0}^{N}B_{(\mu)}\sum_{i=1}^{N}\Big[2^{n_i+1}\sqrt{\mu_i(\mu_i-1)}\,a_i\Psi^*_{(\mu)-2_i}(s)+$$
$$+\sum_{\substack{l=1\\ l\neq i}}^{N}\sqrt{2^{n_i+n_l+2}\mu_i\mu_l}\,a_{il}\Psi^*_{(\mu)-1_i-1_l}(s)\Big]\,ds\Bigg\}$$
$$+(1-S)(\Delta S)^{-1}\sum_{i=1}^{N}\sqrt{2^{n_i+1}}\,m_ib_i\Psi^*_{(m)-1_i}(1)\times$$
$$\times\left\{\int_0^1 s\sum_{(\mu)=0}^{N}B_{(\mu)}\sum_{l=1}^{N}\sqrt{2^{n_l+1}}\mu_lb_l\Psi^*_{(\mu)-1_l}(s)\,ds\right\}E_N^2\,dwy_1\ldots dwy_N. \tag{4}$$

Extending lemma (10) to the N-dimensional case and using E. Ouchar's theorem, it is not difficult to obtain the relation

$$\int_{C^N} \Psi^*_{(m)}(1) \int_0^1 \sum_{(\mu)=0}^{N} B_{(\mu)} \Psi^*_{(\mu)}(s)\, ds\, E_N\, dwy_1 \ldots dwy_N = P_N[x]\Phi_{(m),N}[x] -$$

$$- \int_{C^N} \Bigg\{ 2(1-S)^2(\Delta S)^{-2} \Psi^*_{(m)}(1) \int_0^1 (1-s) \sum_{(\mu)=0}^{N} B_{(\mu)} \sum_{i=1}^{N} \sqrt{2^{n_i+1}} \mu_i b_i \Psi^*_{(\mu)-1_i}(s) \times$$

$$\times \int_1^s \sum_{(\mu)=0}^{N} B_{(\mu)} \sum_{l=1}^{N} \sqrt{2^{n_l+1}} \mu_l b_l \Psi^*_{(\mu)-1_l}(v)\, dv\, ds -$$

$$-(1-S)(\Delta S)^{-1} \Psi^*_{(m)}(1) \int_0^1 (1-s) \sum_{(\mu)=0}^{N} B_{(\mu)} \sum_{i=1}^{N} \Big[2^{n_i+1} \sqrt{\mu_i(\mu_i-1)}\, a_i \Psi^*_{(\mu)-2_i}(s) +$$

$$+ \sum_{\substack{l=1\\ l\neq i}}^{N} \sqrt{2^{n_i+n_l+2} \mu_i \mu_l}\, a_{il} \Psi^*_{(\mu)-1_i-1_l}(s) \Big]\, ds -$$

$$-2(1-S)(\Delta S)^{-1} \sum_{i=1}^{N} \sqrt{2^{n_i+1}}\, m_i b_i \Psi^*_{(m)-1_i}(1) \times$$

$$\times \int_0^1 (1-s) \sum_{(\mu)=0}^{N} B_{(\mu)} \sum_{l=1}^{N} \sqrt{2^{n_l+1}} \mu_l b_l \Psi^*_{(\mu)-1_l}(s)\, ds \Bigg\} E_N\, dwy_1 \ldots dwy_N. \tag{5}$$

Substituting now (3), (4), and (5) into (2), we obtain that $\Phi_{(m),N}[x] = -P_N(x)\Phi_{(m),N}[x]$ in the domain Ω.

From Lemma 2 and Corollary 1 (3) it follows that

Lemma 2. If $P[x(t)] < 0$, $\Omega_h \cup \Gamma_h$ is the closure of the set of functions

$$x_h(t) = \frac{1}{2h} \int_{t-h}^{t+h} x(s)\, ds, \qquad x(t) \in \Omega \cup \Gamma, \qquad x(t) = 0 \quad \text{outside } [0,1],$$

then a functional $V[x(t)]$, satisfying equation (1) in Ω_h and continuous in $\Omega_h \cup \Gamma_h$ for any $h > 0$, cannot attain a positive maximum and a negative minimum in Ω_h.

Corollary. If $P[x(t)] \leq 0$, and the functionals $V_1[x(t)]$ and $V_2[x(t)]$ satisfy equation (1) in Ω and Ω_h, are continuous in $\Omega \cup \Gamma$ and $\Omega_h \cup \Gamma_h$, and $V_1|_\Gamma = V_2|_\Gamma = H[x]$, then $V_1 \equiv V_2$ in Ω.

2. Theorem 1. Suppose a finite-degree functional $G[x(t)]$ is given; $A_{m_1 \ldots m_N}$ are its Fourier-Hermite coefficients; $P[x(t)]$ is a finite-degree functional, $P[x(t)] \leq 0$.

Then in the domain Ω there exists a unique solution of equation (1), coinciding with the given functional $G[x(t)]$ on the surface Γ, which is equal to

$$U[x(t)] = \lim_{N\to\infty} \sum_{m_1,\ldots,m_N=0}^{N} A_{m_1\ldots m_N} \Phi_{m_1\ldots m_N,N}[x(t)] \tag{6}$$

for almost all $x(t) \in \Omega \cup \Gamma$.

Let us now consider the solution of equation (1) in the space $L_2(C)$.

Theorem 2. Let a functional $G[x(t)] \in L_2(C)$ be given; let $A_{m_1\ldots m_N}$ be its Fourier-Hermite coefficients; let $P[x(t)]$ be a functional of finite degree, bounded below almost everywhere, $P[x(t)] \leqslant 0$. Then in the domain Ω there exists a unique solution $U[x(t)] \in L_2(C)$

$$U[x(t)] = \underset{N\to\infty}{\text{L.I.M.}} \sum_{m_1,\ldots,m_N=0}^{N} A_{m_1\ldots m_N}\, \Phi_{m_1\ldots m_N,N}[x(t)] \tag{7}$$

of equation (1), equal to the prescribed functional $G[x(t)]$ on the surface Γ.

The proof of the theorems follows from the properties of the functionals $G[x(t)]$, $P[x(t)]$, Lemma 1, and the corollary.

We note that, putting $P[x(t)] \equiv 0$ in (6) and (7), we obtain the solution of the boundary-value problem for the "Laplace equation."It coincides with the solution given in the author' s paper ([3]), if one computes the multiple Wiener integral of the functional concentrated at a point and then uses the Paley-Wiener formula ([11]).

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CITED LITERATURE

[1] P. Levy, *Problèmes concrets d' analyse fonctionnelle*, Paris, 1951.
[2] E. M. Polishchuk, UMN, **19**, no. 2 (116), 155 (1964).
[3] M. N. Feller, Dop. AN URSR, **12**, 1558 (1965).
[4] M. N. Feller, Dop. AN URSR, **4**, 426 (1966).
[5] N. Wiener, Acta Math., **55**, 117 (1930).
[6] R. E. Graves, Proc. Am. Math. Soc., **4**, 1, 95 (1953).
[7] R. H. Cameron, W. T. Martin, Ann. Math., **48**, 2, 385 (1947).
[8] R. H. Cameron, Proc. Am. Math. Soc., **2**, 6, 944 (1951).

[9] M. Ovchar, Proc. Am. Math. Soc., **3**, 3, 459 (1952).
[10] R. H. Cameron, Ann. Math., **59**, 3, 434 (1954).
[11] H. Wiener, R. Paley, *The Fourier Transform in the Complex Domain*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.