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Abstract

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APPLICATION OF THE CONTOUR-INTEGRAL METHOD TO THE SOLUTION OF MIXED PROBLEMS FOR A PARABOLIC SYSTEM

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In the present note, by the contour-integral method ([1]), the mixed problem (1)–(3) for a system of equations of heat and mass transfer ([2]) is solved. The existence of a solution of this problem is proved, and its representation in the form of the contour integral (21) is given; this integral converges very well for $t > 0$ in comparison with the Laplace integral, owing to the fact that

$$\left|\exp\{\lambda^2 t\}\right| \leq \exp\{-\varepsilon|\lambda|^2 t\}$$

along the contour S. This circumstance makes it possible to compute the contour integral and to construct the solution of the problem effectively ([3]). Special cases of problem (1)–(3) have been solved by other authors ([2]).

1. Consider the mixed problem

$$\partial v/\partial t = A\Delta v; \tag{1}$$

$$\lim_{x\to z}\left\{\left(\alpha_0(z)+\alpha_1(z)\frac{\partial}{\partial t}\right)\frac{\partial v(x,t)}{dn_z}+\left(\beta_0(z)+\alpha_1(z)\beta_1(z)\frac{\partial}{\partial t}\right)v(x,t)\right\}=\psi(z),$$

$$z \in T; \tag{2}$$

$$v(x,0) = \Phi(x), \tag{3}$$

where:

1) A is a constant invertible matrix of second order, composed of elements a_{ij} $(i,j=1,2)$; system (1) is parabolic in the sense of I. G. Petrovsky.

2)

$$B\left(z, \frac{d}{dn_z}, \frac{d}{dt}\right) = \left(\alpha_0(z) + \alpha_1(z)\frac{\partial}{\partial t}\right)\frac{d}{dn_z} + \left(\beta_0(z) + \alpha_1(z)\beta_1(z)\frac{\partial}{\partial t}\right);$$

n_z is the direction of the inner normal to T at the point $z \in T$; $\alpha_k(z)$, $\beta_k(z)$ $(k = 0, 1)$ are matrices of second order, continuous on T; for sufficiently large complex λ,

$$(\alpha_0(z) + \lambda^2\alpha_1(z))^{-1}(\beta_0(z) + \lambda^2\alpha_1(z)\beta_1(z))$$

is bounded by a constant; $\psi(z)$ is a continuous vector-function on T; T is a Lyapunov surface.

3) $\Phi(x)$ is a continuously differentiable vector-function in the three-dimensional domain D, equal to zero in some boundary strip of the domain.

2. Consider the spectral problem

$$A\Delta u - \lambda^2 u = \Phi(x); \tag{4}$$

$$\lim_{x\to z} B\left(z, d/dn_z, \lambda^2\right) u(x, \lambda) = \psi_1(z), \qquad z \in T, \tag{5}$$

in the domain D.

Let p, q be the roots μ of the quadratic equation

$$\mu^2 + (a_{11} + a_{22})\mu + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{6}$$

By virtue of condition 1), the real parts of the complex numbers p, q are negative.

With the aid of the Fourier integral method, a fundamental matrix $P(x - \xi, \lambda)$ of solutions of the homogeneous system (1) is constructed, with a singularity at at the point $x = \xi$. For the elements $P_{ks}(x - \xi, \lambda)$ of the matrix $P(x - \xi, \lambda)$ the following formulas hold:

$$
\begin{aligned}
P_{11}(x-\xi,\lambda) &= \frac{1}{4\pi(p-q)|x-\xi|}\left\{\frac{a_{22}+p}{p}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-p}}\right]\right. \\
&\qquad \left.-\frac{a_{22}+q}{q}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-q}}\right]\right\}, \\
P_{12}(x-\xi,\lambda) &= -\frac{a_{12}}{4\pi(p-q)|x-\xi|}\left\{\frac{1}{p}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-p}}\right]\right. \\
&\qquad \left.-\frac{1}{q}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-q}}\right]\right\}, \\
P_{21}(x-\xi,\lambda) &= -\frac{a_{11}}{4\pi(p-q)|x-\xi|}\left\{\frac{1}{p}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-p}}\right]\right. \\
&\qquad \left.-\frac{1}{q}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-q}}\right]\right\}, \\
P_{22}(x-\xi,\lambda) &= \frac{1}{4\pi(p-q)|x-\xi|}\left\{\frac{a_{11}+p}{p}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-p}}\right]\right. \\
&\qquad \left.-\frac{a_{11}+q}{q}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-q}}\right]\right\},
\end{aligned} \tag{7}
$$

if the roots p, q of the quadratic equation (6) are distinct; here $|x-\xi|$ denotes the length of the vector $x-\xi$; $x = (x_1, x_2, x_3)$, $\xi = (\xi_1, \xi_2, \xi_3)$.

If, however, the roots p, q of the quadratic equation (6) coincide, then for the elements $P_{ks}(x-\xi,\lambda)$ of the matrix $P(x-\xi,\lambda)$ the formulas are

$$
P_{11}(x-\xi,\lambda) = \frac{1}{8\pi p^2}\left\{\lambda\sqrt{-p}\left(1+\frac{a_{22}}{p}\right)+\frac{2a_{22}}{|x-\xi|}\right\}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-p}}\right], \tag{8}
$$

$$
P_{ks}(x-\xi,\lambda) = \frac{a_{ks}}{8\pi p^2}\left\{\frac{\lambda\sqrt{-p}}{|p|}+\frac{2}{|x-\xi|}\right\}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-p}}\right]
$$

for $k = 1,\ s = 2$ or $k = 2,\ s = 1$,

$$
P_{22}(x-\xi,\lambda) = \frac{1}{8\pi p^2}\left\{\lambda\sqrt{-p}\left(1+\frac{a_{11}}{p}\right)-\frac{2a_{11}}{|x-\xi|}\right\}\exp\left[-\lambda\frac{|x-\xi|}{\sqrt{-p}}\right].
$$

3. Denote by $u_1(x,\lambda)$ the solution of the spectral problem (4)–(5) for the corresponding homogeneous system (4). Seeking $u_1(x,\lambda)$ in the form of a simple-layer potential

$$
u_1(a,\lambda) = \iint_T P(x-y,\lambda)\mu(y,\lambda)\,dT_y \tag{9}
$$

leads to the integral equation

$$\mu(z,\lambda) + \iint_T K(z,y,\lambda)\mu(y,\lambda)\,dT_y = \psi_1(z,\lambda), \tag{10}$$

where

$$K(z,y,\lambda) = 2A\left[dP(z-y,\lambda)/dn_z + \left(\alpha_0(z)+\lambda^2\alpha_1(z)\right)^{-1}\left(\beta_0(z)+\lambda^2\alpha_1(z)\beta_1(z)\right)P(z-y,\lambda)\right],$$

$$\psi_1(z,\lambda) = 2A\left(\alpha_0(z)+\lambda^2\alpha_1(z)\right)^{-1}\psi(z).$$

Let R_δ be the region of values of λ satisfying the inequalities

$$\cos\arg\lambda \geq \delta, \qquad |\lambda| \geq R,$$

where R is sufficiently large and δ is a sufficiently small number.

Under conditions 1)–3) of item 1, the kernel $K(z,y,\lambda)$ of the integral equation (10) has a weak singularity; moreover, for $\lambda \in R_\delta$ the estimate

$$|K(z,y,\lambda)| \leq \frac{C}{|z-y|^{2-\alpha}}\exp\{-\varepsilon|\lambda||z-y|\}, \tag{11}$$

holds, where α is the Lyapunov exponent, and C, ε are positive constants.

With the aid of estimate (11) one proves

Theorem 1. *Under conditions 1)–3) of Sec. 1, the spectral problem (4)–(5) has a solution $u_1(x,\lambda)$, analytic in $\lambda \in R_\delta$, representable in the form of the double-layer potential (9), where $\mu(y,\lambda)$ is the solution of the integral equation (10), for the resolvent $R(z,y,\lambda)$ of which an estimate of the form (11) is valid*

$$|R(z,y,\lambda)| \leq \frac{C}{|z-y|^{2-\alpha}}\exp\{-\varepsilon|\lambda|\,|z-y|\}. \tag{12}$$

If D_1 is a domain lying together with its boundary in the domain D, then for all $x \in \overline{D}_1$ the estimate

$$\left|\frac{\partial^k u_1(x,\lambda)}{\partial x_i^k}\right| \leq \frac{C}{\sigma^{1+k}}\exp\{-\varepsilon|\lambda|\sigma\} \quad (k=0,1,2), \tag{13}$$

holds, where σ is the distance between the boundaries of the domains D, D_1. For all $x \in D+T$ the inequality

$$\left|\frac{d^k}{dn_z^k} u_1(x,\lambda)\right| \leq C \quad (k=0,1) \tag{14}$$

holds.

Let $Q(x,\xi,\lambda)$ be the regular part of the Green matrix $G(x,\xi,\lambda)$ of problem (4) –(5):

$$G(x,\xi,\lambda) = P(x-\xi,\lambda) - Q(x,\xi,\lambda). \tag{15}$$

Seeking $Q(x,\xi,\lambda)$ in the form of a single-layer potential

$$Q(x,\xi,\lambda) = \iint_T P(x-y,\lambda)\mu(y,\xi,\lambda)\,dT_y, \tag{16}$$

we arrive at the integral equation:

$$\mu(z,\xi,\mu) + \iint_T K(z,y,\lambda)\mu(y,\xi,\lambda)\,dT_y = f(z,\xi,\lambda), \tag{17}$$

where

$$f(z,\xi,\lambda) = 2A\left[\frac{d}{dn_z} + \left(\alpha_0(z)+\lambda^2\alpha_1(z)\right)^{-1}\left(\beta_0(z)+\lambda^2\alpha_1(z)\beta_1(z)\right)\right]P(z-\xi,\lambda).$$

As is seen, the integral equations (10) and (17) differ from each other only in the free terms $\psi_1(z,\lambda)$, $f(z,\xi,\lambda)$. Consequently,

$$\mu(z,\xi,\lambda) = f(z,\xi,\lambda) - \iint_T R(z,y,\lambda)f(y,\xi,\lambda)\,dT_y. \tag{18}$$

From (7), (8), (11), (12), (16), and (18) it follows that

Theorem 2. *Under conditions 1)–3) of Sec. 1, for all $\lambda \in R_\delta$ there exists a solution analytic in λ*

$$u_2(x,\lambda,\Phi) = -\iiint_D G(x,\xi,\lambda)\Phi(\xi)\,dD_\xi; \tag{19}$$

the regular part $Q(x,\xi,\lambda)$ of the Green matrix is determined by formula (16). For every pair of points x,ξ from the domain D_1, lying together with its boundary in the domain D, the estimate

$$\left|\frac{\partial^k Q(x,\xi,\lambda)}{\partial x_i^k}\right| \le \frac{C}{\sigma^{k+3}} \exp\{-\varepsilon|\lambda|\,|x-\xi|\} \quad (k=0,1,2). \tag{20}$$

holds. For all $x \in D+T$ lying on the normal n_z $(z \in T)$, and $\xi \in D_1$, the inequality

$$|dQ(x,\xi,\lambda)/dn_z| \le C/\sigma^2$$

is satisfied.

4. Let S be an infinite open contour of the λ-plane, coinciding with the boundary of the domain R_δ outside a circle of sufficiently large radius centered at the origin. Following the proof scheme of Theorems 38, 39 of the author' s book (2), it is not difficult to prove

Theorem 3. *Under conditions 1)–3) of item 1, the mixed problem (1)–(3) has a solution $v(x,t)$, representable by the formula*

$$v(x,t) = \frac{1}{\pi\sqrt{-1}} \int_S e^{\lambda^2 t} \left\{ \frac{u_1(x,\lambda)}{\lambda} - \lambda \iiint_D G(x,\xi,\lambda)\Phi(\xi)\, dD_\xi \right\} d\lambda. \tag{21}$$

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Note: Figure translations are in progress. See original paper for figures.