The theory of characteristic vectors and its application to the study of the asymptotic behavior of solutions of differential systems. I

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Abstract

Full Text

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THE THEORY OF CHARACTERISTIC VECTORS AND ITS APPLICATION TO THE STUDY OF THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENTIAL SYSTEMS

In the present work, we introduce the concept of the characteristic vector, which generalizes the notions of the characteristic exponent [?] and the characteristic degree [?]. It is proved that any linear system of order n possesses no more than n distinct characteristic vectors. The characteristic exponent can be viewed as the first component of the characteristic vector.

If the characteristic exponent is zero, we examine the second component of the characteristic vector to investigate stability. If the second components of the characteristic vectors are negative, the linear system is "regular"in a certain sense, and the nonlinear perturbations satisfy specific conditions, it is established that the trivial solution of the nonlinear system will be asymptotically stable. In cases where the second component of the characteristic vector is also zero, we proceed to consider its third component, and so on.

Consequently, the theory of characteristic vectors provides an unlimited iterative process for investigating stability in critical cases. The fundamental results of this work were briefly reported in [?].

1. Definitions and Basic Properties

Let $x(t)$ be an arbitrary function. Let us assume that there exists the limit:

$$a_0 = \lim_{t\to\infty} \frac{\ln \|x(t)\|}{t}$$

Then $\|x(t)\| \leq ae^{a_0 t}$, where $a = \text{const}$. Furthermore, let us assume that there exists a finite limit:

$$a_1 = \lim_{t\to\infty} \frac{\ln(\|x(t)\| e^{-a_0 t})}{\ln t}$$

Then $\|x(t)\| \leq a_2 e^{a_0 t}(\ln t)^{a_1+\epsilon}$. In general, let us assume there exists a finite limit:

$$a_k = \lim_{t\to\infty} \frac{\ln(\|x(t)\| e^{-a_0 t} \prod_{i=1}^{k-1} (\ln_i t)^{-a_i})}{\ln_k t}$$

where $\ln_k t$ denotes the k-th iterated logarithm.

Suppose that m is a sufficiently large positive integer. By definition, we shall call the vector $\vec{a}^{(m)}(x) = (a_0, a_1, \dots, a_m)$ the characteristic vector of the m-th order of the function $x(t)$. The set of characteristic vectors $\{\vec{a}^{(m)}\}$ is ordered lexicographically: $\vec{a}^{(m)} > \vec{b}^{(m)}$ if there exists some index j such that $a_j > b_j$ and $a_i = b_i$ for all $i < j$.

2. Linear Systems

We consider the linear system:

$$\dot{x} = A(t)x \quad (3.1)$$

where $A(t)$ is a piecewise continuous real matrix defined for $0 < t < \infty$. We assume the system satisfies the boundedness condition:

$$\|A(t)\| \leq K_A = \text{const} \quad (3.2)$$

The set $\{\alpha^{(m)}\}$ of m-th order characteristic vectors of the solutions to (3.1) is bounded. We observe the following properties: a) The existence of characteristic vectors for solutions of (3.1) is guaranteed under (3.2). b) $\alpha^{(m)}(x_1 + x_2) \leq \max(\alpha^{(m)}(x_1), \alpha^{(m)}(x_2))$.

These properties demonstrate that the set $\{\alpha^{(m)}\}$ of characteristic vectors satisfies the conditions of a Lyapunov norm [**?**]. From [**?**], we conclude that the n-dimensional vector space of solutions to system (3.1) has no more than n distinct characteristic vectors. Note that for $m = 0$, we obtain the characteristic exponent [**?**], and for $m = 1$, we obtain the characteristic degree [**?**].

3. Adjoint Systems and Regularity

Consider the adjoint system:

$$\dot{y} = -yA(t) \quad (5.1)$$

Let $\alpha^{(m)}$ and $\alpha^{*(m)}$ be the characteristic vectors of (3.1) and (5.1), respectively. If $X(t)$ is the fundamental matrix of (3.1), then $Y(t) = X^{-1}(t)$ consists of rows that are solutions to (5.1). It can be shown that for corresponding vectors:

$$\alpha_i^{(m)} + \alpha_i^{*(m)} \geq 0 \quad (5.2)$$

The set of characteristic vectors Γ generates an abelian ordered group [**?**]. Following the theorems of R. E. Vinograd [**?**], we define regularity as follows: **Definition.** System (3.1) is called regular of the m-th order if:

$$\alpha_i^{(m)} + \alpha_i^{*(m)} = 0 \quad \text{for all } i = 0, \dots, m.$$

4. Necessary and Sufficient Conditions for Regularity

Theorem 1. System (3.1) is regular of order m if and only if:

$$\sum_{i=1}^{n} \alpha_i^{(m)} = \lim_{t\to\infty} \text{vec} \left(\int_{t_0}^{t} \text{Tr } A(\tau) d\tau \right)$$

where the limit is understood in the sense of the components of the characteristic vector.

Corollary 1. A necessary condition for the regularity of order m is the existence of the finite limits defining the characteristic vector for the determinant of the fundamental matrix.

5. Stability of Nonlinear Systems

Consider the perturbed system:

$$\frac{dx}{dt} = A(t)x + f(t,x) \quad (10.2)$$

where $f(t,x)$ satisfies $\|f(t,x)\| \leq c(t)\|x\|^{1+\sigma}$ for some $\sigma > 0$.

Theorem 3. Suppose that: 1) System (3.1) is regular of order m. 2) The characteristic vectors of the linear part satisfy $\alpha_i^{(m)} < 0$. 3) The perturbation satisfies $|c(t)| \leq K(\ln_m t)^{-1-\epsilon}$. Then the trivial solution of (10.2) is asymptotically stable, and there exists a family of solutions satisfying:

$$\|x(t)\| \leq C(\ln_m t)^{-\delta} \quad (10.4)$$

Theorem 4. If system (3.1) is regular and has k characteristic vectors with negative components and $n-k$ characteristic vectors with positive components, then there exists a k-dimensional manifold S containing the origin such that solutions starting on S tend to zero as $t \to \infty$. If $f(t,x)$ is analytic, S is an analytic manifold.

6. Instability Theorems

Theorem 6. Consider system (10.2). Suppose that: 1) For the linear system (3.1), there exists at least one characteristic vector $\alpha^{(m)} > 0$. 2) The perturbation $f(t,x)$ is bounded such that $\|f(t,x)\| \leq c(t)\gamma(\|x\|)$. Then the trivial solution of (10.2) is unstable.

The stability region in the space of characteristic vectors can be viewed as the interior of a half-space defined by the lexicographical ordering. The theory of characteristic vectors provides a more granular refinement of the stability criteria compared to the classical method of characteristic exponents.

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Note: Figure translations are in progress. See original paper for figures.