ERGODIC THEOREMS FOR GENERAL DYNAMICAL SYSTEMS

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ERGODIC THEOREMS FOR GENERAL DYNAMICAL SYSTEMS

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Consider a semigroup X with a σ-ring of measurable sets $\mathfrak{B}$ and a space with a complete σ-finite measure $(\Omega, \mathfrak{S}, m)$. A **dynamical system** in Ω with time from X will mean a collection $(\Omega, \mathfrak{S}, m, T_x,\ x \in X)$, where $T_x,\ x \in X$, is a family of transformations of the space Ω having the properties: 1) if $f(\omega)$ is a measurable real-valued function on Ω, then $f(T_x\omega)$ is a measurable function on $X \otimes \Omega$; 2) $T_{x_1}T_{x_2} = T_{x_1x_2}$ for any $x_1, x_2 \in X$; 3) $m(T_x^{-1}\Lambda) = m(\Lambda)$, if $x \in X,\ \Lambda \in \mathfrak{S}$ ($T_x^{-1}\Lambda$ is the full inverse image of the set Λ under the transformation T_x; measurability of the set $T_x^{-1}\Lambda$ follows from condition 1)).

An important place in the theory of dynamical systems with real time is occupied by Birkhoff' s ergodic theorem, which asserts that for any integrable function $f(\omega)$ the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(T_x\omega)\, dx.$$

exists almost everywhere. Neumann' s ergodic theorem asserts that for any function with integrable square this limit exists in the sense of convergence in mean square. The present note is devoted to extending the ergodic theorems of Birkhoff and Neumann to general dynamical systems.

In what follows $(\Omega, \mathfrak{S}, m, T_x, X)$ is a fixed dynamical system; B is a Banach space and $|\cdot|$ is the norm in B; $L_B^p,\ 1 \le p < \infty$, is the space of all measurable B-valued functions on Ω for which

$$\int_\Omega |f|^p\, dm < \infty;$$

$I_B^p \subset L_B^p$ is the subspace of functions invariant with respect to all transformations $T_x,\ x \in X$; $\mathfrak{S}_A \subset \mathfrak{S}$ is the subalgebra of sets invariant with respect to all transformations $T_x,\ x \in A$; $\mathfrak{S} = \mathfrak{S}_X$; R and Z are the additive groups of real and integer numbers; R_+ and Z_+ are the corresponding semigroups of nonnegative numbers.

1. In this section we consider averaging of "motions" over generalized sequences of sets A_n, $n \in N$ (N is an arbitrary ordered set). Denote: $x^{-1}D = \{y : xy \in D\}$, $Dx^{-1} = \{y : yx \in D\}$. We assume that: 1) $x^{-1}D \in \mathfrak{B}$, $Dx^{-1} \in \mathfrak{B}$, if $D \in \mathfrak{B}$; 2) on $\mathfrak{B}$ there exists a σ-finite measure μ such that $\mu(Dx^{-1}) \leq \mu(D)$ for any $x \in X$, $D \in \mathfrak{B}$; 3) A_n and A_{nx} ($n \in N$, $x \in X$) are measurable and $0 < \mu(A_n) < \infty$; moreover, on the sequence A_n, $n \in N$, the following conditions will be imposed as needed:

(E_1). For any $x \in X$, $D \in \mathfrak{B}$

$$\lim_{n \in N} \frac{\mu(A_n \cap D) - \mu(A_n \cap x^{-1}D)}{\mu(A_n)} = 0.$$

(E_2). Monotonicity. $A_{n'} \subseteq A_{n''}$, if $n' < n''$.

(E_3). There exists a constant $0 < C_1 < \infty$ such that, for any $n \in N$, $y \in X$, we have

$$\mu^*\{x : \mu(A_{nx} \cap A_{ny}) > 0\} \leq C_1 \mu(A_n)$$

(here μ^* is the outer measure induced by the measure μ).

(E_4). There exists a sequence of measurable sets M_n, $n \in N$, such that

$$\lim_{n \in N} \frac{\mu^*(A_{nM}n)}{\mu(M_n)} = C_2 < \infty.$$

Let us note that condition (E_1) follows from the condition

$$(\mathrm{E}_1'). \quad \lim_{n \in N} \frac{\mu(A_n \Delta x^{-1} A_n)}{\mu(A_n)} = 0 \quad \text{for every } x \in X.$$

If X is a locally compact group and μ is its right Haar measure, then condition (E_3) is equivalent to the following condition:

$$(\mathrm{E}_3'). \quad \mu\{x : \mu(A_{nx} \cap A_n) > 0\} \leq C_1 \mu(A_n) \quad \text{for every } n \in N.$$

This condition is satisfied if

$$(\mathrm{E}_3''). \quad \mu^*(A_n^{-1} A_n) \leq C_1 \mu(A_n) \quad \text{for every } n \in N.$$

If the group X is unimodular, or if the sets A_n are symmetric ($A_n^{-1} = A_n$), then condition (E_4) also follows from (E_3''), and moreover $C_2 \leq C_1$.

Example 1. If A is an arbitrary measurable set in R^m, star-shaped with respect to the origin and with finite Lebesgue measure ($0 < \mu(A) < \infty$), then the sequence of similar sets $A_t = tA$, $0 < t < \infty$, satisfies conditions $(\mathrm{E}_1)-(\mathrm{E}_4)$, with $C_1 = \mu(A + (-A))/\mu(A)$, $C_2 = 1$. Any generalized sequence of bounded convex sets $A_n \subset R^m$, $n \in N$, satisfies condition (E_3''), and hence also (E_4); if the sets A_n are symmetric, $C_1 = 2^m$; for arbitrary convex sets $C_1 < m!2^m$. Let $r(A)$ be the supremum of the radii of balls contained in the set A; if $\lim_{n\in N} r(A_n) = \infty$, then the sequence of bounded convex sets A_n, $n \in N$, satisfies condition (E_1') (see (2)). If a sequence $A_n \subset R_+^m$ satisfies conditions $(\mathrm{E}_1)-(\mathrm{E}_4)$ in R^m, then it satisfies them in R_+^m. In all these cases the sets M_n can be chosen so that $C_2 = 1$.

Example 2. In Z^l, the properties $(\mathrm{E}_1)-(\mathrm{E}_4)$ with $C_1 = 2^l$, $C_2 = 1$ are possessed, for example, by the sequences of sets

$$A_n = \{z = (z_1, \dots, z_l) : |z_i| \le r_n,\ i = 1, \dots, l\}, \quad B_n = \left\{ z : \sum_{i=1}^{l} z_i^2 \le r_n^2 \right\},$$

if $r_n \to \infty$; the sequences $A_n \cap Z_+^l$ and $B_n \cap Z_+^l$ possess the properties $(\mathrm{E}_1)-(\mathrm{E}_4)$ in Z_+^l.

Example 3. In a compact group K we satisfy the requirements $(\mathrm{E}_1)-(\mathrm{E}_4)$ by taking $A_n = K$ ($C_1 = C_2 = 1$).

Example 4. If $X = X^{(1)} \times \dots \times X^{(k)}$ is the direct product of measurable semigroups and the sequences $A_n^{(i)} \subset X^i$, $n \in N$, possess the properties $(\mathrm{E}_1)-(\mathrm{E}_4)$ with constants $C_1^{(i)}$ and $C_2^{(i)}$, then the sequence

$$A_n = A_n^{(1)} \times \dots \times A_n^{(k)}$$

in X possesses these properties, and one may put

$$C_1 = \prod_{i=1}^{k} C_1^{(i)}, \qquad C_2 = \prod_{i=1}^{k} C_2^{(i)}.$$

Hence there follows, for example, the existence of sequences of sets with the properties $(\mathrm{E}_1)-(\mathrm{E}_4)$ in Abelian groups of compact origin: every such group is isomorphic to a group of the form $K \times R^m \times Z^l$.

Theorem 1. *Let the sequence of sets $A_1, A_2, \dots$ satisfy conditions $(\mathrm{E}_2)-(\mathrm{E}_4)$; let $f(\omega)$ be an arbitrary R_+-valued measurable function on Ω;*

$$f^*(\omega) = \sup_{1\le n<\infty} \frac{1}{\mu(A_n)} \int_{A_n} f(T^x\omega)\, \mu(dx).$$

Then the inequalities

$$m\{\omega : f^*(\omega) > a\} \leq \begin{cases} \dfrac{C_1C_2}{a} \displaystyle\int_\Omega f\,dm, \\ \dfrac{2C_1C_2}{a} \displaystyle\int_{\{\omega: f(\omega) > a/2\}} f\,dm, \end{cases} \qquad (a > 0); \tag{1}$$

hold.

$$\int_\Omega f^*\,dm \leqslant 2\left(m(\Omega) + C_1C_2 \int_\Omega f \log^+ f\,dm\right); \tag{2}$$

$$\int (f^*)^p\,dm \leqslant \begin{cases} \dfrac{2^pC_1C_2}{p-1} \displaystyle\int_\Omega f^p\,dm, & (1 < p < \infty), \\ 2^p\left(1 + \dfrac{pC_1C_2}{1-p}\right)(m(\Omega))^{1-p} \displaystyle\int_\Omega f\,dm, & (0 < p < 1). \end{cases} \tag{3}$$

Theorem 2. If the generalized sequence $A_n,\ n \in N$, satisfies conditions (E_1)–(E_4), then for any function $f \in L_B^p,\ 1 \leq p < \infty$, almost everywhere there exists the limit

$$\lim_{n \in N} \frac{1}{\mu(A_n)} \int_{A_n} f(T_x\omega)\,\mu(dx) = \hat{f}(\omega); \tag{4}$$

if $f \in L_B^p,\ 1 < p < \infty$, or $f \in L_B^1$ and $m(\Omega) < \infty$, then the limit (4) exists also in the sense of convergence in L_B^p; the space L_B^p can be represented in the form $L_B^p = I_B^p \oplus M_B^p$, where M_B^p is the subspace generated by the functions $f(T_x\omega) - f(\omega)$ $(f \in L_B^p,\ x \in X)$; $\hat{f}$ is the projection of f onto I_B^p.

Thus, relation (4) determines $\hat{f}$ from f uniquely (of course, up to equivalence), independently of the choice of the averaging sequence of sets $A_n,\ n \in N$. From Theorem 2 it also follows easily that for any set $\Lambda \in \mathfrak{F}$ with $m(\Lambda) < \infty$ the equality

$$\int_\Lambda f\,dm = \int_\Lambda \hat{f}\,dm$$

holds; in particular, if $m(\Omega) = 1$, then $\hat{f} = M(f \mid \mathfrak{F})$.

Theorems 1 and 2 generalize the results of Wiener ([8]), Pitt ([6]), Calderon ([1]), and others.

2. In what follows X is a locally compact semigroup; $\mathfrak{B}$ is the σ-ring of Borel sets in X; $q(dx)$ is a normalized Borel measure; q^{*k} is the k-fold convolution of the measure q.* We assume that the measure q is not concentrated on any proper subsemigroup of the semigroup X. More precisely, if $D \in \mathfrak{B},\ q(D) = 1$, and

$$\overline{D} = \bigcup_{k=1}^{\infty} D^k,$$

then the lower measure

$$q_*(x^{-1}\overline{D}) = 1$$

for every $x \in X$. If X has an identity e, this assumption can be weakened by setting

$$\overline{D} = \bigcup_{k=1}^{\infty} (D \cup D^{-1})^k,$$

where

$$D^{-1} = \{x: \ x \in X,\ e \in Dx\}.$$

Theorem 3. If

$$\nu_n = \frac{1}{n}\sum_{k=1}^{n} q^{*k}$$

and $f \in L_B^p,\ 1 \le p < \infty$, then almost everywhere there exists the limit:

$$\lim_{n\to\infty} \int_X f(T_x\omega)\,\nu_n(dx) = \hat{f}(\omega); \tag{5}$$

if $f \in L_B^p,\ 1 < p < \infty$, or $f \in L_B^1$ and $m(\Omega) < \infty$, then the limit exists also in the sense of convergence in L_B^p; the function $\hat{f}$ is the projection of f onto I_B^p; if f is an R_+-valued measurable function on Ω and

$$f^*(\omega) = \sup_{1\le n\le\infty} \int_X f(T_x\omega)\,\nu_n(dx),$$

then inequalities (1)–(3) are valid, with $C_1 = C_2 = 1$.

* For the definition of convolution of Borel measures on a locally compact semigroup, see ([5]).

The proof is based on applying to the operator $Sf = \int_X f(T_x\omega)q(dx)$ the L_B^p ergodic theorem of Dunford–Schwartz (4). For $X = Z_+,\ q(\{1\}) = 1,\ q(Z_+ \setminus \{1\}) = 0$, Theorem 3 assumes the classical form of Birkhoff' s individual ergodic theorem.

Theorem 4. Let $p_k(dx),\ k = 1, 2, ...,$ be a sequence of normalized Borel measures on a locally compact group X; $\bar{p}_k(dx) = p_k(dx^{-1})$; $\nu_n = p_1 * \cdots * p_n * \bar{p}_n * \cdots * \bar{p}_1$. If $f \in L_B^p$, $1 < p < \infty$, or $m(\Omega) < \infty$, $f \in L_B^1$ and

$$\int_\Omega |f| \log^+ |f|\, dm < \infty,$$

then almost everywhere on Ω there exists the limit (5), and moreover $\hat{f} \in L_B^p$; for any function $f \in L_B^p$, $1 < p < \infty$, or $f \in L_B^1$, $m(\Omega) < \infty$, the limit (5) exists in the sense of convergence in L_B^p.

In the proof one uses the limit theorems of Dub (3) and Rota (7). From Theorems 3 and 4 there follows easily

Corollary 1. Let $m(\Omega) < \infty$; X be a locally compact group; the measure q have support $c(q)$ and be symmetric: $q(dx) = q(dx^{-1})$; $\nu_n = q^{*n}$. If the function f satisfies the corresponding conditions of Theorem 4, the limit (5) exists almost everywhere and in L_B^p if and only if $\mathfrak{I}_{c(q)\cdot c(q)} = \mathfrak{I}_{c(q)}$; the function $\hat{f}$ is the projection of f onto I_B^p.

For another approach to this result see (9).

Theorem 5. Let the semigroup X be separable; $\xi_1, \xi_2, \ldots$ a stationary sequence of random variables with values in X;

$$\nu_n(A) = \frac{1}{n} \sum_{k=1}^{n} P(\xi_1 \ldots \xi_k \in A) \quad (A \in \mathfrak{B});$$

if $f \in L_B^p$, $1 < p < \infty$, or $m(\Omega) < \infty$, $f \in L_B^1$ and

$$\int_\Omega |f| \log^+ |f| \, dm) < \infty,$$

then the limit (5) exists almost everywhere; if $f \in L_B^p$, $1 < p < \infty$, or $m(\Omega) < \infty$ and $f \in L_B^1$, the limit (5) exists in the sense of convergence in L_B^p.

Let us note that if in Theorems 4 and 5 the measures ν_n satisfy the condition

$$\lim_{n \to \infty} (\nu_n(D) - \nu_n(x^{-1}D)) = 0$$

for all $x \in X$, $D \in \mathfrak{B}$, then $\hat{f}$ coincides with the projection of f onto I_B^p.

The following theorem makes it possible to enlarge the class of "averaging" sequences of measures found in Theorems 2-5.

Theorem 6. Let λ_n and ν_n, $n \in N$, be generalized sequences of normalized measures on $\mathfrak{B}$, where: 1) $\lambda_n \ll \nu_n$ $(n \in N)$; 2) there exists a constant $C < \infty$ such that

$$\frac{d\lambda_n}{d\nu_n}(x) < C \quad (x \in X,\ n \in N);$$

3)

$$\lim_{n \in N} (\lambda_n(D) - \lambda_n(x^{-1}D)) = 0$$

for all $D \in \mathfrak{B}$, $x \in X$; if for any $f \in L_B^p$, $1 \leq p < \infty$, almost everywhere there exists

$$\lim_{n \in N} \int_X f(T_x\omega)\nu_n(dx) \in L_B^p,$$

then almost everywhere there exists the limit

$$\lim_{n\in N}\int_X f(T_x\omega)\lambda_n(dx),$$

coinciding with the projection of f onto L_B^p.

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Note: Figure translations are in progress. See original paper for figures.