THE RADIATIVE TRANSFER EQUATION IN A MODEL OF ISOTROPIC POINT SCATTERERS

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Abstract

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THE RADIATIVE TRANSFER EQUATION IN A MODEL OF ISOTROPIC POINT SCATTERERS

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The radiative transfer equation [1–3] is the fundamental equation for solving direct and inverse problems in the spectroscopy of dispersing substances [4]. However, up to the present time it has been derived phenomenologically from considerations of the energy balance of radiation. It is of interest to give a statistical derivation of the transfer equation within the framework of the theory of multiple scattering of waves on a statistical ensemble of scatterers [5–9].

We shall restrict ourselves to consideration of the transfer equation for a scalar (sound) field

$$\mathbf{s}\frac{\partial J(\mathbf{rs})}{\partial r} = -\varkappa(\mathbf{r})J(\mathbf{rs}) + \int p(\mathbf{ss}'|\mathbf{r})J(\mathbf{rs}')\,ds', \tag{1}$$

where $J(\mathbf{rs})$ is the radiation intensity at the point $\mathbf{r}$ in the direction of the unit vector $\mathbf{s}$; $\varkappa(\mathbf{r})$ is the volume extinction coefficient; $p(\mathbf{ss}'|\mathbf{r})$ is the volume differential scattering coefficient; ds' is an element of solid angle.

Let us analyze the derivation of equation (1) on the simplest model of point isotropic uncorrelated scatterers. The scattering operator of an isolated point isotropic scatterer is specified by the equality [5] $u_j(\mathbf{r}) = fG_0(\mathbf{r}-\mathbf{r}_j)\Phi(\mathbf{r}_j)$, where $u_j(\mathbf{r})$ is the scattered field at the observation point $\mathbf{r}$; f is the isotropic scattering amplitude;

$$G_0(\mathbf{r}-\mathbf{r}_j) = \frac{\exp ik_0|\mathbf{r}-\mathbf{r}_j|}{|\mathbf{r}-\mathbf{r}_j|}$$

is the Green function of the homogeneous medium surrounding the scatterer; $\Phi(\mathbf{r}_j)$ is the incident field at the center of the scatterer $\mathbf{r}_j$. We note that a point isotropic scatterer can be realized in the form of a gas bubble in a liquid, whose radius is small compared with the wavelength of the incident field. Let $\psi(\mathbf{r})$ denote the total field, equal to the sum of the incident field and the field multiply scattered on an ensemble of scatterers. We are interested in the configuration averages $\langle\psi(\mathbf{r})\rangle$ and $\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle$. For them, in [5] the equations

$$[\Delta + k_0^2 + 4\pi f\rho(\mathbf{r})]\langle\psi(\mathbf{r})\rangle = 0, \tag{2}$$

$$\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle = \langle\psi(\mathbf{r})\rangle\langle\psi^0(\mathbf{r}_0)\rangle + \int \rho(\mathbf{r}_j)\, d\mathbf{r}_j\, |f|^2 G(\mathbf{r}\mathbf{r}_j) G^*(\mathbf{r}_0\mathbf{r}_j)\langle|\psi(\mathbf{r}_j)|^2\rangle, \tag{3}$$

were obtained, where $\rho(\mathbf{r}_j)$ is the density of the scatterers, and $G(\mathbf{r}\mathbf{r}_j)$ is the Green function of equation (2).

Let us turn to equation (2). Introduce the effective complex index of refraction

$$m \equiv 1 + (2\pi\rho/k_0^2)f = n + iq,$$

where $n = 1 + (2\pi\rho/k_0^2)\,\mathrm{Re}\, f$ is the real refractive index; $q = \varkappa/2k_0$; $\varkappa = (4\pi\rho/k_0)\,\mathrm{Im}\, f$ is the volume extinction coefficient. We assume that m differs little from unity, so that the quantities $\delta n = n - 1$ and q are small. We need to find the general solution of equation (2), depending on arbitrary functions, as well as the Green function of this equation. We shall solve the equation by the WKB method, assuming the density of scatterers

$\rho(\mathbf{r})$ a slowly varying function. The method of finding the general solution in a region free of caustics is well known (see, for example, [10]). It consists in substituting $\langle\psi\rangle = A \exp ik_0\tau$, where A is the amplitude and τ is the eikonal, with subsequent expansion $A = A_0 + (ik_0)^{-1}A_1 + \ldots$. The presence of a small imaginary part in the refractive index m causes no fundamental difficulties. The general solution has the form

$$\langle\psi(\mathbf{r})\rangle \sim A(\mathbf{r}^0)(d\Sigma^0/d\Sigma)^{1/2}\left[ik_0|\mathbf{r}-\mathbf{r}^0| - \frac{1}{2}\int_0^{|\mathbf{r}-\mathbf{r}^0|}\chi(\mathbf{r}^0 + \mathbf{s}^0 l)\, dl\right]. \tag{4}$$

In (4) it is assumed that at the points $\mathbf{r}^0$ of the surface of some initial wave front Σ^0 the amplitude $A(\mathbf{r}^0)$ and the eikonal $\tau(\mathbf{r}^0)$, which is set equal to zero, are specified. Integration with respect to dl is performed along the straight ray emerging from the point $\mathbf{r}^0$ in the direction $\mathbf{s}^0 = (\mathbf{r}-\mathbf{r}^0)/|\mathbf{r}-\mathbf{r}^0|$ and arriving at the observation point $\mathbf{r}$ of the wave front Σ. By $d\Sigma^0$ and $d\Sigma$ are denoted the area elements of the surfaces of the named wave fronts in neighborhoods of the points $\mathbf{r}^0$ and $\mathbf{r}$, cut out by an elementary ray tube. In deriving formula (4) we have neglected retardation and refraction effects.

The Green' s function of the Helmholtz equation with a slowly varying real refractive index was calculated in [11]. Using the method of that work, we obtain

$$G(\mathbf{r}\mathbf{r}_j) \sim |\mathbf{r}-\mathbf{r}_j|^{-1} \exp\left[ik_0|\mathbf{r}-\mathbf{r}_j| - \frac{1}{2}\int_0^{|\mathbf{r}-\mathbf{r}_j|} \chi\left(\mathbf{r}_j + \frac{\mathbf{r}-\mathbf{r}_j}{|\mathbf{r}-\mathbf{r}_j|} l\right) dl\right]. \quad (5)$$

Let us turn to equation (3). Substitute into it the mean field (4), the Green' s function (5), and write its solution in the form of an iteration series

$$\begin{aligned}\langle\psi(\mathbf{r})\psi^*(\mathbf{r}_0)\rangle = \langle\psi(\mathbf{r})\rangle\langle\psi^*(\mathbf{r}_0)\rangle + \\ + \sum_{n\geq 1}\int \rho(\mathbf{r}_n)\ldots\rho(\mathbf{r}_1)\, d\mathbf{r}_n \ldots d\mathbf{r}_1\, G(\mathbf{r}\mathbf{r}_n) f G^*(\mathbf{r}_0\mathbf{r}_n) f^* |G(\mathbf{r}_n\mathbf{r}_{n-1})|^2 |f|^2 \ldots \\ \ldots |G(\mathbf{r}_2\mathbf{r}_1)|^2 |f|^2 |\langle\psi(\mathbf{r}_1)\rangle|^2.\end{aligned} \quad (6)$$

First put in (6) $\mathbf{r}_0 = \mathbf{r}$ and introduce into each n-th term of the sum the unit factor

$$\begin{aligned}\int \delta(\mathbf{s}_1 - \mathbf{s}^0)\, d\mathbf{s}_1 \delta(\mathbf{r}^0 + l_0\mathbf{s}_1 - \mathbf{r}_1)\, d(\mathbf{r}^0 + l_0\mathbf{s}_1)\delta(\mathbf{r}^0 + l_0\mathbf{s}_1 + l_1\mathbf{s}_2 - \mathbf{r}_2)\, d(l_1\mathbf{s}_2)\ldots \\ \ldots \delta(\mathbf{r}^0 + l_0\mathbf{s}_1 + \ldots + l_n\mathbf{s}_{n+1} - \mathbf{r})\, d(l_n\mathbf{s}_{n+1}),\end{aligned} \quad (7)$$

where integration is performed over all indicated differentials. After this, by means of identical transformations, we arrive at the result

$$\langle|\psi(\mathbf{r})|^2\rangle = \int I(\mathbf{r}\mathbf{s})\, d\mathbf{s}, \quad (8)$$

$$I(\mathbf{r}\mathbf{s}) = \int |A(\mathbf{r}^0)|^2\, d\Sigma^0 \int_0^\infty dl\, F(\mathbf{r}\mathbf{s}, \mathbf{r}^0\mathbf{s}^0, l). \quad (9)$$

In (9) the integration is performed over the surface of the initial wave front and over the length of the rays emerging from its points and arriving at the observation point $\mathbf{r}$ in the specified direction $\mathbf{s}$. The function F satisfies an integral equation which, after differentiation with respect to l, turns into the integro-differential equation

$$\left(\frac{\partial}{\partial l} + \mathbf{s}\frac{\partial}{\partial \mathbf{r}}\right) F(\mathbf{r}\mathbf{s}, \mathbf{r}^0\mathbf{s}^0, l) = -\chi(\mathbf{r})F(\mathbf{r}\mathbf{s}, \mathbf{r}^0\mathbf{s}^0, l) + \int p(\mathbf{s}\mathbf{s}' \mid \mathbf{r})F(\mathbf{r}\mathbf{s}', \mathbf{r}^0\mathbf{s}^0, l)\, d\mathbf{s}' \quad (10)$$

with the supplementary condition

$$F(\mathbf{rs}, \mathbf{r}^0\mathbf{s}^0, 0) = \delta(\mathbf{r} - \mathbf{r}^0)\delta(\mathbf{s} - \mathbf{s}^0). \tag{11}$$

The coefficient $p(\mathbf{ss}' \mid \mathbf{r}) = |f|^2\rho(\mathbf{r})$.

Analogously to formula (8), one can obtain the relation

$$\langle\vec{\Pi}(\mathbf{r})\rangle \equiv (k_0 c_0 \rho_0/4i)\langle\psi^*\nabla\psi - \psi\nabla\psi^*\rangle = \int \mathbf{s}J(\mathbf{rs})\, d\mathbf{s}, \tag{12}$$

if series (6) is differentiated with respect to $\mathbf{r}$ and $\mathbf{r}_0$, with the passage to the limit $\mathbf{r}_0 \to \mathbf{r}$. In (12), $\vec{\Pi}(\mathbf{r})$ denotes the vector of the energy flux in the sound wave, c_0 and ρ_0 are the sound velocity and density of the medium surrounding the scatterers, and the function $J = (c_0\rho_0 k_0^2/2)I$.

To clarify the physical meaning of the function $J(\mathbf{rs})$, multiply equation (10) by $|A(\mathbf{r}^0)|^2$ and integrate over $d\Sigma^0$ and dl within the limits (9). After integration, for the function $\widetilde{J} = J - J^0$, where $J^0(\mathbf{rs}) = \frac{1}{2}c_0\rho_0 k_0^2 \times |\langle\psi(\mathbf{r})\rangle|^2\delta(\mathbf{s} - \mathbf{s}^0)$, we obtain the transport equation (1) with an additional term on the right-hand side, equal to the integral of the product $p(\mathbf{ss}'|\mathbf{r})J^0(\mathbf{rs}')$ over $d\mathbf{s}'$. Comparison of relation (12) with the corresponding relation of the phenomenological theory of radiative transfer shows that the introduced function $J(\mathbf{rs})$ may be regarded as the radiation intensity. In the terminology of (3), J^0 represents the intensity of the direct radiation, and $\widetilde{J}$ that of the scattered radiation. The intensity of the scattered radiation determines the fluctuations of the total field and of the energy flux.

Let us make several remarks concerning the derivation of the transport equation (1). Equation (1) can be obtained without recourse to the iterative series (6), by solving equation (3) with the aid of substitution (8), (9) and an analogous substitution for the squared modulus of the mean field.

The definition of the radiation intensity by the surface integral (9) is convenient in scattering problems when the source of the incident field is located outside the scattering medium. However, in radiation problems with sources inside the medium one should use another representation of the intensity, in the form of a volume integral

$$\widetilde{J}(\mathbf{rs}) = \frac{c_0\rho_0 k_0^2}{2}\int_0^\infty dl \int G(\mathbf{rs}, \mathbf{r}'\mathbf{s}', l)\, |\langle\psi(\mathbf{r}')\rangle|^2\, d\mathbf{r}', \tag{13}$$

where the function $G(\mathbf{rs}, \mathbf{r}'\mathbf{s}', l)$ again satisfies equation (10), but now with the boundary condition $G(\mathbf{rs}, \mathbf{r}'\mathbf{s}', 0) = \delta(\mathbf{r} - \mathbf{r}')p(\mathbf{ss}'|\mathbf{r}')$.

Let us briefly discuss the conditions of applicability of the transport equation (1) within the framework of the model under consideration. These conditions have already been formulated qualitatively in the text. The conditions are as follows:

the wavelength $\lambda_0 = 2\pi/k_0$ is large in comparison with the scatterer radius a; the correction to the real part $\delta n \sim \lambda_0^2 a\rho$ of the complex refractive index m and its imaginary part $q \sim \lambda_0/l_e$, where $l_e = \chi^{-1}$ is the extinction length, are small in comparison with unity; the scatterer density $\rho(\mathbf{r})$ is a slowly varying function of the coordinates. The last restriction on the density presupposes the smallness of quantities of the order: $(\delta n)(l_e/l_\rho)^2$, $(\delta n)(\lambda_0 l_e/l_\rho^2)$, $q(l_e/l_\rho)^2$, where $l_\rho \sim \rho/|\nabla\rho|$ is the spatial scale of variation of the density. We note that, owing to the smallness of the factors δn and q, the ratio l_e/l_ρ need not necessarily be small in comparison with unity.

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Note: Figure translations are in progress. See original paper for figures.