Two-point and multi-point Taylor formulas

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Abstract

A formula is considered by which the value of the function $f(x) \in C^n$ on a given interval is determined through its values and the values of its derivatives at m points ($m \geq 2$) of the given interval. The considered examples demonstrate the possibility of effectively using the obtained generalized Taylor formula for solving boundary value problems. Bibliography 3.

Full Text

Preamble

This work, following the developments in [1], investigates the properties of the function $h(x)$, defined as $h(x) = g(x+1) - g(x)$. We consider a smooth function $f(t)$ such that $f(t)f(1-t)dt$ is normalized over the interval $(0, 1)$. Specifically, we define the transition functions $g_n(x)$ and $h_n(x)$ which belong to the class C^n and satisfy specific boundary conditions at the nodes x_1 and x_2.

The kernel $K(x)$ is constructed using the auxiliary function $g_n(x)$ as follows:

$$K(x) = g_n\left(1 - \{|x| - 1 + |x|\}\right)$$

For $n = 1, 2, 3$, the functions $h_n(x)$ are elements of $C^n(-1, +1)$ and satisfy $h_n(0) = 1$, while their derivatives vanish at the boundaries: $h_n^{(i)}(0) = h_n^{(i)}(\pm 1) = 0$. These functions are essential for constructing local approximations that maintain high degrees of smoothness across sub-intervals.

Local Approximation and Error Estimates

Let $f(x)$ be a function in C^{n+1} on the interval $[x_1, x_2]$. We define a local interpolant using the basis functions $h_n(x)$ and the values of the function and its derivatives at the endpoints. The approximation formula is given by:

$$f(x) \approx \sum_{i=0}^{n} \frac{f^{(i)}(x_1)}{i!}(x - x_1)^i h_n\left(\frac{x - x_1}{h}\right) + \sum_{i=0}^{n} \frac{f^{(i)}(x_2)}{i!}(x - x_2)^i h_n\left(\frac{x - x_2}{h}\right)$$

where $h = x_2 - x_1$. The remainder term R_{n+1} for this approximation can be estimated as:

$$|R_{n+1}| \leq \frac{M}{(n+1)!} h^{n+1}$$

where $M = \max |f^{(n+1)}(\xi)|$ for $\xi \in [x_1, x_2]$. This estimate ensures that the approximation converges as the mesh size h decreases, provided the function $f(x)$ possesses sufficient smoothness.

Application to Boundary Value Problems

The proposed method is particularly effective for solving linear differential equations of the form $Lu = f$ with boundary conditions at $x = a$ and $x = b$. We represent the solution $u(x)$ as a linear combination of the basis functions $\phi_{i,m}(x, a)$ and $\phi_{i,m}(x, b)$. These basis functions are constructed to satisfy the homogeneous boundary conditions, allowing the global solution to be assembled from local components.

For a boundary value problem on $[a, b]$, we partition the interval into k sub-intervals with nodes $x_i = a + ih$. In each sub-interval (x_{i-1}, x_i), the solution is approximated by:

$$u(x) \approx \sum \alpha_i \Phi_i(x)$$

where the coefficients α_i are determined by matching the values and derivatives of the solution at the internal nodes. This approach leads to a system of linear algebraic equations.

Numerical Examples

To demonstrate the effectiveness of the method, we consider a fourth-order differential equation:

$$(x + 21)y^{IV} + gy - kx = 0$$

with boundary conditions $y(0) = \alpha_1$, $y'(0) = 0$, and $y(1) = 0$, $y'(1) = 0$. Using the basis functions $h_4(x)$ and a partition of the interval $[0, 1]$ with $h = 0.2$, we obtain a numerical solution. The coefficients c_1 and c_2 are determined by solving the resulting system of equations.

In a second example, we approximate the function $y = \sin x$ on a given interval. The numerical results at the nodes x_1, x_2, x_3 show high agreement with the exact values: - At x_1: $y_{calc} = 0.4933$, $y_{exact} = 0.5000$ - At x_2: $y_{calc} = 0.7084$, $y_{exact} = 0.7033$ - At x_3: $y_{calc} = 0.8606$, $y_{exact} = 0.8659$

These results confirm that the method provides a robust framework for both function approximation and the numerical solution of differential equations, maintaining high accuracy with a relatively small number of nodes.

Note: Figure translations are in progress. See original paper for figures.