The problem of transverse vibrations of a viscoelastic rod

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Abstract

This paper considers the problem of free transverse vibrations of a finite viscoelastic rod where one end is fixed and the other is free. Mathematically, the problem is formulated as seeking a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial^4 u}{\partial x^4} + b\frac{\partial^5 u}{\partial x^4 \partial t} = 0 \quad (0 < x < l,\, 0 < t < T) \tag{1}$$

under the initial conditions:

$$u|_{t=0} = \varphi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \quad (0 < x < l) \tag{2}$$

and the boundary conditions:

$$u|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad (t \geq 0), \tag{3}$$

$$\left.\begin{aligned} \left. a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^3 u}{\partial x^2 \partial t}\right|_{x=l} = 0 \\ \left. a\frac{\partial^3 u}{\partial x^3} + b\frac{\partial^4 u}{\partial x^3 \partial t}\right|_{x=l} = 0 \end{aligned}\right\} \quad (t > 0), \tag{4}$$

where a and b are physical constants, and $u(x,t)$ represents the deflection function of the rod axis.

It is proved that under certain restrictions imposed on the functions $\varphi(x)$ and $\psi(x)$, a solution to the problem (1)-(4) exists and can be represented as a contour integral:

$$u(x,t) = \frac{1}{2\pi i} \lim_{\nu \to \infty} \int_{\Gamma_\nu} u(x,\lambda)\lambda e^{\lambda^2 t}\, d\lambda, \tag{5}$$

where $y(x,\lambda)$ is the solution to a specific auxiliary problem, and Γ_ν ($\nu = 1, 2, 3, ...$) is a sequence of expanding closed contours. By calculating the contour integral (5), $u(x,t)$ is represented as a series. It is further proved that the resulting solution is unique within a certain class of functions and is stable. To obtain and mathematically justify the solution to problem (1)-(4), the residue method and the contour integral method developed by M. L. Rasulov are applied. Additionally, the solution to the problem is constructed using the Fourier method.

Illustrations: 1. Bibliography: 8.

Full Text

Preamble

This work continues the investigations initiated in [1] and [2] regarding the dynamics of structural elements. We consider the boundary value problem for the following partial differential equation:

$$\frac{\partial^4 u}{\partial x^4} + a\frac{\partial u}{\partial t} + b\frac{\partial^5 u}{\partial x^4 \partial t} + \frac{\partial^2 u}{\partial t^2} = 0, \quad (0 < x < l,\ 0 < t < T) \tag{1.1}$$

subject to the initial conditions:

$$u(x,0) = \phi(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad (0 < x < l) \tag{1.2}$$

and the boundary conditions for a clamped beam:

$$u(0,t) = 0, \quad \frac{\partial u(0,t)}{\partial x} = 0, \quad (t > 0) \tag{1.3}$$

$$u(l,t) = 0, \quad \frac{\partial^2 u(l,t)}{\partial x^2} = 0, \quad (t > 0). \tag{1.4}$$

Here, $a > 0$ and $b > 0$ are physical constants, and $u(x,t)$ represents the displacement function. In Sections 3 and 4, we analyze the solution using the contour integral method. The solution can be represented in the form:

$$u(x,t) = -\frac{1}{2\pi i}\int_\Gamma y(x,\lambda)\lambda e^{\lambda t} d\lambda \tag{1.5}$$

where $y(x,\lambda)$ is the solution to the corresponding spectral problem.

Section 2. Separation of Variables

Applying the method of separation of variables $u(x,t) = X(x)T(t)$ to equations (1.1)-(1.4), we obtain the following equation for the temporal component $T(t)$:

$$T'' + (b\lambda^4 + a)T' + \lambda^4 T = 0 \tag{2.1}$$

The spatial component $X(x)$ must satisfy the differential equation:

$$X^{IV} - \lambda^4 X = 0 \tag{2.2}$$

with the boundary conditions:

$$X(0) = 0,\ X'(0) = 0,\ X''(l) = 0,\ X'''(l) = 0. \tag{2.3}$$

The general solution to (2.2) is given by:

$$X(x) = A\cosh\lambda x + B\sinh\lambda x + C\cos\lambda x + D\sin\lambda x \tag{2.4}$$

Substituting (2.4) into the boundary conditions (2.3) leads to the characteristic equation $\cosh\alpha\cos\alpha = -1$, where $\alpha = \lambda l$. The roots α_k of this equation are well-known: $\alpha_1 = 1.875$, $\alpha_2 = 4.694$, and for $n > 3$, $\alpha_n \approx \frac{\pi}{2}(2n-1)$. The corresponding eigenfunctions $X_k(x)$ are defined on $[0,l]$ as:

$$X_k(x) = (\sinh\alpha_k - \sin\alpha_k)(\cosh\frac{\alpha_k x}{l} - \cos\frac{\alpha_k x}{l}) - (\cosh\alpha_k + \cos\alpha_k)(\sinh\frac{\alpha_k x}{l} - \sin\frac{\alpha_k x}{l}) \tag{2.7}$$

These eigenfunctions satisfy the orthogonality property:

$$\int_0^l X_k^2(x)dx = \frac{l}{4}(\sinh\alpha_k - \sin\alpha_k)^2 \tag{2.10}$$

The general solution to the original problem (1.1)-(1.4) can then be expressed as a series:

$$u(x,t) = \sum_{k=1}^{\infty} X_k(x)[C_k \cosh q_k t + D_k \sinh q_k t]e^{-p_k t} \tag{2.11}$$

where p_k and q_k are determined by the parameters a, b and the eigenvalues λ_k.

Section 3. Spectral Analysis and Green' s Function

To justify the solution, we consider the transformed problem:

$$\frac{d^4y}{dx^4} + \lambda^2 y + (a + b\lambda^2)\frac{d^4y}{dx^4} = f(x,\lambda) \tag{3.1}$$

with boundary conditions $y(0) = y'(0) = y''(l) = y'''(l) = 0$. The function $f(x,\lambda)$ is defined as:

$$f(x,\lambda) = \lambda\phi(x) + \psi(x) + b\phi^{IV}(x) \tag{3.3}$$

The solution to this boundary value problem can be expressed using the Green' s function $G(x,\xi,\lambda)$:

$$y(x,\lambda) = \int_0^l G(x,\xi,\lambda)f(\xi,\lambda)d\xi \tag{3.4}$$

The Green' s function is constructed as:

$$G(x,\xi,\lambda) = \frac{A(x,\xi,\lambda)}{\Delta(\lambda)} \tag{3.10}$$

where $\Delta(\lambda)$ is the characteristic determinant:

$$\Delta(\lambda) = 2(1 + \cosh\alpha z l \cosh\beta z l) \tag{3.9}$$

Here, the parameters α, β, z are related to the coefficients of the original equation. We analyze the asymptotic behavior of the Green's function in the complex λ-plane to ensure the convergence of the contour integral (1.5).

Section 5. Convergence and Final Solution

By evaluating the residues of the integrand in (1.5) at the poles λ_{mk}, we obtain the final form of the solution. The poles are determined by the roots of the characteristic equation and the quadratic form (2.1). Specifically, we find:

$$\lambda_{mk} = -p_k \pm q_k \tag{5.5}$$

The resulting series representation for $u(x,t)$ is:

$$u(x,t) = \sum_{k=1}^{\infty} \left[\frac{\cosh q_k t - \frac{p_k}{q_k} \sinh q_k t}{\|X_k\|^2} \int_0^l X_k(\xi)\phi(\xi)d\xi + \frac{\sinh q_k t}{q_k \|X_k\|^2} \int_0^l X_k(\xi)\psi(\xi)d\xi \right] e^{-p_k t} \tag{5.12}$$

This solution accounts for the damping effects introduced by the parameters a and b. In the limiting case where $b = 0$, the solution reduces to the standard vibration model for a beam with internal friction. The convergence of this series and its derivatives is guaranteed by the smoothness assumptions on the initial functions $\phi(x)$ and $\psi(x)$, as discussed in Sections 3 and 4. The results demonstrate that the contour integral method provides a robust framework for solving non-classical problems in structural mechanics.

Note: Figure translations are in progress. See original paper for figures.

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