ON CLASSES OF REGULAR FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract

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ON CLASSES OF REGULAR FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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The author has introduced ([1,2]) the classes $Q_D,\ M_D,\ N_D$, which constitute an essential generalization of univalent, starlike univalent, and convex univalent functions in the disk* to the case of several complex variables. The essential generalization here consists in the fact that, on passing to one variable, one obtains functions which, generally speaking, are not univalent (see Remark 1). But in the presence of a certain weight one obtains univalent functions, as follows from the definitions of the classes $Q_D,\ M_D,\ N_D$.

In the present note four more classes of functions are introduced, two of which are an essential generalization of functions of bounded rotation and almost convex functions to the case of several complex variables, and a number of results are given for all the enumerated classes of functions of several complex variables, in particular for the last four classes. In fact, the whole exposition is carried out in the case of two complex variables, since in the case of n variables it is carried out analogously.

1. Let D be a bounded complete bicircular domain with center at the point $(0,0)$, and let k_0 and k_0' be fixed finite numbers from the whole set of complex numbers. Suppose that in the domain D the function $F(w,z)$, $F(0,0)=1$, is regular. Consider the set $D\cap\{z=k_0w\}$ $(D\cap\{w=k_0'z\})$, which is the section of the domain D by the plane $z=k_0w$ $(w=k_0'z)$.

Definition 1. We shall say that in the section $D\cap\{z=k_0w\}$ $(D\cap\{w=k_0'z\})$ the function $wF(w,z)$ $(zF(w,z))$ is a **function of bounded rotation**, and the function $wF(w,z)-w$ $(zF(w,z)-z)$ is with **contracted distortion**, if the function $wF(w,k_0w)$ $(zF(k_0'z,z))$, as a function of one complex variable w (z) in the corresponding disk ([1,2]), is a function of bounded rotation**, and the function** $wF(w,k_0w)-w$ $(zF(k_0'z,z)-z)$ **is with contracted distortion***.

Definition 2. Denote by V_D (Y_D) the class of functions $F(w,z)$, $F(0,0)=1$, regular in the domain D, possessing the following properties: 1) in the section of the domain D by each plane from all possible analytic planes $z=kw$**** the function $wF(w,z)$ $(wF(w,z)-w)$ is a function of bounded rotation (with contracted distortion); 2) in the section $D\cap\{w=0\}$ the function

$zF(0,z)$ $(zF(0,z)-z)$ is a function of bounded rotation (with contracted distortion)*****.

Let in the domain D the function $F(w,z)$, $F(0,0)=1$, be regular. Then from properties 1) and 2) of the class $V_D(Y_D)$ it follows: a) in the section of the domain D by each

* Regular functions; the same below.
** That is, the function $wF(w,k_0w)$ $(zF(k_0'z,z))$ satisfies in the corresponding disk the condition $\mathrm{Re}(wF(w,k_0w))'_w>0$ $(\mathrm{Re}(zF(k_0'z,z))'_z>0)$, or the equivalent condition $|\arg(wF(w,k_0w))'_w|<\pi/2$ $(|\arg(zF(k_0'z,z))'_z|<\pi/2)$.
*** That is, the function $wF(w,k_0w)-w$ $(zF(k_0'z,z)-z)$ satisfies in the corresponding disk the condition $|(wF(w,k_0w)-w)'_w|<1$ $(|(zF(k_0'z,z)-z)'_z|<1)$.
**** k runs through the whole set of complex numbers, except ∞.
***** In the case of one variable this is the class of functions regular in the disk $D\{|z|<R\}$, $F(z)$, $F(0)=1$, possessing the following property: in D the function $zF(z)$ $(zF(z)-z)$ is a function of bounded rotation (with contracted distortion). Similarly in the case of the classes Q_D, M_D, N_D.

by a plane from all possible analytic planes $w=k'z$ * the function $zF(w,z)$ $(zF(w,z)-z)$ is a function with bounded rotation (with contracted distortion); b) in the section $D\cap\{z=0\}$ the function $wF(w,0)$ $(wF(w,0)-w)$ is a function with bounded rotation (with contracted distortion), and conversely. Thus, properties 1), 2) and a), b) are equivalent.

Theorem 1. In order that a function $F(w,z)$, $F(0,0)=1$, regular in the domain D, belong to the class V_D (Y_D), it is necessary and sufficient that in D

$$\mathrm{Re}\,L_1[F(w,z)]>0 \qquad (|L_1[F(w,z)]-1|<1),$$

where

$$L_1[F(w,z)]\equiv F(w,z)+wF'_w(w,z)+zF'_z(w,z).$$

The proof is analogous to the proof of the theorem in ([1]).

2. Let $S_D(0)$ be the class of functions $F(w,z)$ regular in the domain D, $F(0,0)=0$, for which $|F(w,z)|<1$ in D, and let $C_D(1)$ be the class of functions $F(w,z)$ regular in D, $F(0,0)=1$, for which $\mathrm{Re}\,F(w,z)>0$ in D.

The following proposition is obvious:

I. If a function $F(w,z)\in S_D(0)$, then the function

$$(1+F(w,z))\,(1-F(w,z))^{-1}\equiv\Phi(w,z)\in C_D(1),$$

and conversely, if a function $\Phi(w,z)\in C_D(1)$, then the function

$$(\Phi(w,z)-1)(\Phi(w,z)+1)^{-1}\equiv F(w,z)\in S_D(0).$$

The criteria for membership of regular functions in the classes M_D, N_D, V_D, Y_D (see ([1]) and Theorem 1 of the present note), the integral formulas (1), (2) from ([3]) (formula (1) is taken with $m = 1$) and Proposition I make it possible to establish relations among functions of the classes $M_D, N_D, V_D, Y_D, S_D(0), C_D(1)$ (14 theorems, besides Proposition I, of which we shall state the 3 theorems needed below).

Theorem 2. If a function $\Phi(w,z) \in N_D$, then the function $L_1[\Phi(w,z)] \equiv F(w,z) \in M_D$, and conversely, if a function $F(w,z) \in M_D$, then the function

$$\int_0^1 F(\varepsilon w, \varepsilon z)\, d\varepsilon = \Phi(w,z) \in N_D \ **.$$

Theorem 3. If a function $\Phi(w,z) \in V_D$, then the function $L_1[\Phi(w,z)] \equiv F(w,z) \in C_D(1)$, and conversely, if a function $F(w,z) \in C_D(1)$, then the function

$$\int_0^1 F(\varepsilon w, \varepsilon z)\, d\varepsilon = \Phi(w,z) \in V_D.$$

Theorem 4. If a function $\Phi(w,z) \in Y_D$, then the function $L_1[\Phi(w,z)] - 1 \equiv F(w,z) \in S_D(0)$, and conversely, if a function $F(w,z) \in S_D(0)$, then the function

$$1 + \int_0^1 F(\varepsilon w, \varepsilon z)\, d\varepsilon = \Phi(w,z) \in Y_D.$$

3. **Definition 3.** A function $F(w,z)$, $F(0,0) = 1$, regular in the domain D, will be called a function close to functions of the class N_D (M_D), if there exists a function $\Phi(w,z) \in N_D(M_D)$ such that in D

$$\operatorname{Re}\left(\frac{L_1[F(w,z)]}{L_1[\Phi(w,z)]}\right) > 0 \qquad \left(\operatorname{Re}\left(\frac{F(w,z)}{\Phi(w,z)}\right) > 0\right).$$

The set of all such functions will be denoted by $R_D(P_D)$.

Between functions of the classes R_D and P_D there is the same relation as between functions of the classes N_D and M_D (Theorem 2).

Theorem 5. If a function $F(w,z) \in Q_D$, then in D $F(w,z) \neq 0$.

Theorem 6. Every function belonging to the class R_D is a function of the class Q_D.

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* k' runs through the whole set of complex numbers, except ∞.

** ε is real; likewise below.

Theorem 7. $M_D \subset R_D$.

Theorem 8. $Y_D \subset V_D \subset R_D$.

Theorem 9. The class P_D is not contained in the class Q_D and does not coincide with it.

Theorem 10. $V_D \subset C_D(1) \subset P_D$.

In proving Theorems 5-10, use is made of the definitions of the classes Q_D, M_D, N_D, R_D, P_D, the theorem from (1), Theorems 1, 2 of the present note, and the integral formula (1) (for $m = 1$) from (3).

We also note that $N_D \subset M_D \subset Q_D$.

Remark 1. Let us point out one peculiarity of functions of the classes $Q_D, M_D, N_D, V_D, Y_D, R_D$: on passing to one variable we obtain functions which, generally speaking, are not univalent. Indeed, the functions $(1 - w - z)^{-1}$, $[1 - (w + z)^2]^{-1}$ (here, for simplicity, D is the hypercone $\{|w| + |z| < 1\}$), by virtue of the theorem from (1), belong to the class M_D and, consequently, to the classes R_D and Q_D. At the same time, in the disk $|z| < 1$ the function $(1-z)^{-1}$ is univalent, while the function $(1-z^2)^{-1}$ is not univalent. Introducing for the function $(1 - z^2)^{-1}$ the weight (factor z), we also obtain a univalent function in $|z| < 1$. It is easy to construct similar examples also in the case of the classes N_D, V_D, Y_D.

4. Let us indicate, in the classes V_D, Y_D, R_D, P_D, estimates for the basic functionals (as before in analogous cases). Let $\overline{D}_r = r\overline{D}$, where r is a positive number less than one. Bearing in mind Theorem 6 of this note, we arrive at the proposition:

II. If the function $F(w, z) \in R_D$, then in $\overline{D}_r$ the estimates (3), (4) from (2) hold.

Theorem 11. If the function $F(w, z) \in P_D$, then in $\overline{D}_r$

$$(1 + r)^{-3}(1 - r) \leq |F(w, z)| \leq (1 - r)^{-3}(1 + r).$$

Theorem 12. If the function $F(w, z) \in V_D$, then in $\overline{D}_r$

$$2r^{-1}\ln(1 + r) - 1 \leq \operatorname{Re} F(w, z) \leq -2r^{-1}\ln(1 - r) - 1,$$

$$2r^{-1}\ln(1 + r) - 1 \leq |F(w, z)| \leq -2r^{-1}\ln(1 - r) - 1,$$

$$|\operatorname{Im} F(w, z)| \leq -r^{-1}\ln(1 - r^2),$$

and for $\operatorname{Re} L_1[F(w, z)]$, $|L_1[F(w, z)]|$, $|\operatorname{Im} L_1[F(w, z)]|$ the corresponding estimates (for $p = 1$) established in Theorem 6 of the article (4) hold.

Theorem 13. If the function

$$F(w,z) = 1 + \sum_{k=p\geq 1}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right) \in Y_D,$$

then in $\overline{D}_r$

$$1 - (p+1)^{-1} r^p \leq \operatorname{Re} F(w,z) \leq 1 + (p+1)^{-1} r^p,$$

$$1 - (p+1)^{-1} r^p \leq |F(w,z)| \leq 1 + (p+1)^{-1} r^p,$$

$$1 - r^p \leq \operatorname{Re} L_1[F(w,z)] \leq 1 + r^p,$$

$$1 - r^p \leq |L_1[F(w,z)]| \leq 1 + r^p.$$

Theorem 14. If the function

$$F(w,z) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right) \in R_D \ (P_D),$$

then for $k > 0$

$$A_k(D) \leq (k+1)^2, \qquad B_k(D) \leq k+1$$

$$\left(A_k(D) \leq (k+1)^4, \qquad B_k(D) \leq (k+1)^2\right),$$

where

$$A_k(D) \equiv \sup_{(w,z)\in D} \sum_{l=0}^{k} |a_{k-l,l}|^2 |w|^{2(k-l)} |z|^{2l},$$

$$B_k(D) \equiv \sup_{(w,z)\in D} \left| \sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right|.$$

Theorem 15. If the function

$$F(w,z) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l \right) \in V_D(Y_D),$$

then for $k > 0$ the sharp estimates hold

$$A_k(D) \leq 4(k+1)^{-2}, \qquad B_k(D) \leq 2(k+1)^{-1}$$

$$(A_k(D) \leq (k+1)^{-2}, \qquad B_k(D) \leq (k+1)^{-1}).$$

In proving Theorems 11-15, Proposition II and Theorems 2-4 (Theorem 2 in the case of the classes R_D and P_D) of this note, Theorems 1, 2, 6 from ([4]), the integral formula (1) (for $m = 1$) from ([3]), and the estimate of Taylor coefficients for almost convex functions ([5]) are used.

5. Everything set forth in this note, except for the estimates $A_k(D)$, is completely preserved also in the case of a bounded complete circular domain K with center at the point $(0, 0)$.

6. Consequences of the estimates $A_k(D)$ in Theorems 14, 15 are estimates of the Taylor coefficients of functions of the classes R_D, P_D, V_D, Y_D, and in the classes V_D, Y_D these estimates are sharp. Finally, with respect to Proposition II, Theorems 11-13, and the consequences just mentioned, there is a remark of the same character as in ([2]), item 5.

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