SIMPLE EXAMPLES OF UNDECIDABLE ASSOCIATIVE CALCULI

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Abstract

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SIMPLE EXAMPLES OF UNDECIDABLE ASSOCIATIVE CALCULI

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1. A method for constructing an associative calculus with an undecidable equivalence problem was first indicated in papers of A. A. Markov ([1]) and Post ([2]). The defining systems of calculi constructed by this method consist of a considerable number of relations. G. S. Tseitin, in paper ([3]), constructed an associative calculus with a defining system of 7 relations for which the equivalence problem is undecidable. A method for constructing an associative calculus with a defining system of 7 relations and an undecidable equivalence problem is also indicated by Scott ([4]).

In the present paper we construct an associative calculus with an undecidable equivalence problem for a fixed word, whose defining system consists of 5 relations. In addition, we indicate a method for constructing an associative calculus with a defining system of 3 relations for which the equivalence problem for some fixed word is also undecidable.

2. Let A denote the alphabet $\{a, b, c, d, e\}$. Let i be a nonnegative integer. Denote by $\mathfrak{C}_i$ the following associative calculus in the alphabet A:

$$\begin{gathered} ac \leftrightarrow ca, \qquad ad \leftrightarrow da, \\ bc \leftrightarrow cb, \qquad bd \leftrightarrow db, \\ ce \leftrightarrow eca, \qquad de \leftrightarrow edb, \\ cd^i ca \leftrightarrow cd^i cae. \end{gathered}$$

By $\mathfrak{E}_i$ we denote the associative calculus in the alphabet A with defining system

$$\begin{gathered} ac \leftrightarrow ca, \qquad ad \leftrightarrow da, \\ bc \leftrightarrow cb, \qquad bd \leftrightarrow db, \\ ce \leftrightarrow eca, \qquad de \leftrightarrow edb, \\ cd^i ca \leftrightarrow cd^i ce. \end{gathered}$$

G. S. Tseitin, in paper ([3]), proved that, whatever the nonnegative integer i, for the calculus $\mathfrak{C}_i$ the equivalence problem for some fixed word is undecidable. In an analogous way one can prove that, whatever the nonnegative integer i, for the calculus $\mathfrak{E}_i$ the equivalence problem for some fixed word is undecidable.

3. Denote by B the alphabet $\{\alpha, \sigma\}$, and by φ the following normal algorithm in the alphabet $A \cup B$:

$$\begin{aligned} a &\to \alpha\alpha\sigma\alpha, \\ b &\to \alpha\sigma\alpha\alpha, \\ c &\to \alpha\sigma, \\ d &\to \sigma\sigma, \\ e &\to \alpha\alpha\alpha. \end{aligned}$$

Denote by $\mathfrak{H}$ the following associative calculus in the alphabet B:

$$\begin{gathered} \alpha\sigma\alpha\alpha\sigma \leftrightarrow \sigma\sigma\alpha\alpha\sigma\alpha, \\ \alpha\alpha\sigma\alpha\sigma\alpha \leftrightarrow \sigma\sigma\alpha\alpha\alpha\sigma\alpha, \\ \alpha\sigma\alpha\alpha\alpha\sigma \leftrightarrow \alpha\sigma\sigma\alpha\sigma\alpha, \\ \sigma\sigma\sigma\alpha\alpha\sigma\sigma\alpha\alpha\sigma\alpha \leftrightarrow \sigma\sigma\sigma\alpha\sigma\sigma\alpha\alpha\alpha\alpha, \\ \alpha\alpha\alpha\alpha\sigma\sigma\alpha\alpha\sigma\alpha \leftrightarrow \sigma\sigma\alpha\alpha\alpha\alpha. \end{gathered}$$

Theorem 1. *Whatever the words P and Q in the alphabet A, in order that $\mathfrak{C}_1 : P \parallel Q$, it is necessary and sufficient that $\mathfrak{H} : \varphi(P) \parallel \varphi(Q)$.*

Necessity follows from the fact that

$$\begin{array}{ll} \mathfrak{H}: \ \varphi(ac) \parallel \varphi(ca), & \mathfrak{H}: \ \varphi(ad) \parallel \varphi(da), \\ \mathfrak{H}: \ \varphi(bc) \parallel \varphi(cb), & \mathfrak{H}: \ \varphi(bd) \parallel \varphi(db), \\ \mathfrak{H}: \ \varphi(ce) \parallel \varphi(eca), & \mathfrak{H}: \ \varphi(de) \parallel \varphi(edb), \end{array}$$

$$\mathfrak{H}: \ \varphi(cdca) \parallel \varphi(cdce).$$

Sufficiency is proved by induction on the length of the $\mathfrak{H}$-sequence connecting the words $\varphi(P)$ and $\varphi(Q)$.

The **base of the induction** is obvious.

The **induction step** is carried out on the basis of the following lemma.

Lemma 1. *Whatever the word R in the alphabet A and the word V in the alphabet B, if $\mathfrak{H} : \varphi(R) \parallel V$, then one can construct a word S in the alphabet A such that $\mathfrak{C}_1 : R \parallel S$ and $\varphi(S) = V$.*

It follows immediately from Theorem 1 that for the calculus $\mathfrak{H}$ the equivalence problem for a certain fixed word is undecidable.

4. Denote by B the alphabet $A \cup \{f_1, \dots, f_7\}$, and by $\mathfrak{K}$ the following associative calculus in the alphabet B:

$$\begin{array}{ll} f_1 \leftrightarrow ac, & f_1 \leftrightarrow ca, \\ f_2 \leftrightarrow ad, & f_2 \leftrightarrow da, \\ f_3 \leftrightarrow bc, & f_3 \leftrightarrow cb, \\ f_4 \leftrightarrow bd, & f_4 \leftrightarrow db, \\ f_5 \leftrightarrow ce, & f_5 \leftrightarrow ef_1, \\ f_6 \leftrightarrow de, & f_6 \leftrightarrow ef_4, \\ f_7 \leftrightarrow cf_1, & f_7 \leftrightarrow f_7e, \\ f_7 \leftrightarrow cf_1, & f_7 \leftrightarrow f_7e \end{array}$$

(the last two relations are repeated because in what follows we shall need the fact that the number of relations of the calculus $\mathfrak{K}$ is a power of the number 2).

Lemma 2. *Whatever the words P and Q in the alphabet A, $\mathfrak{C}_0 : P \parallel Q$ if and only if $\mathfrak{K} : P \parallel Q$.*

Denote by Γ the alphabet $\{\beta, \gamma\}$, and by ψ the following normal algorithm in the alphabet $B \cup \Gamma$:

$$f_1 \to \beta\beta\gamma\beta\gamma^{12},$$

$$\dots\dots\dots\dots$$

$$f_i \to \beta\beta\gamma^i\beta\gamma^{13-i},$$

$$\dots\dots\dots\dots$$

$$f_{12} \to \beta\beta\gamma^{12}\beta\gamma,$$

where $f_8, \dots, f_{12}$ denote the letters a, b, c, d, e, respectively.

Denote by $A_1, \dots, A_{16}, B_1, \dots, B_{16}$ words in the alphabet B such that the system $A_i \leftrightarrow B_i$ $(i = 1, \dots, 16)$ is a defining system of the calcul-

calculus $\mathfrak{K}$. Denote by C_i the word $\psi(A_i)$, and by D_i the word $\psi(B_i)$ $(i = 1, \dots, 16)$. Denote by $\mathfrak{L}$ the associative calculus in the alphabet Γ whose defining system is the system $C_i \leftrightarrow D_i$ $(i = 1, \dots, 16)$.

Lemma 3. *Whatever the words P and Q in the alphabet B, $\mathfrak{K} : P \parallel Q$ if and only if $\mathfrak{L} : \psi(P) \parallel \psi(Q)$.*

Denote by $p_{i,j}$ the i-th letter from the left of the word C_j ($i = 1, \dots, 16^*$, $j = 1, \dots, 16$), and by $q_{i,j}$ the i-th letter from the left of the word D_j ($i = 1, \dots, 32^{**}$, $j = 1, \dots, 16$). Denote by M the word

$$p_{1,1} \cdots p_{1,16} p_{2,1} \cdots p_{2,16} \cdots p_{16,1} \cdots p_{16,16},$$

and by N the word

$$q_{1,1} \cdots q_{1,16} q_{2,1} \cdots q_{2,16} \cdots q_{32,1} \cdots q_{32,16}.$$

Denote by Δ the alphabet $\Gamma \cup \{\varepsilon\}$, and by $\mathfrak{M}$ the following associative calculus in the alphabet Δ:

$$\varepsilon\beta\beta \leftrightarrow \beta\varepsilon, \qquad \varepsilon\gamma\beta \leftrightarrow \beta\varepsilon,$$

$$\varepsilon\beta\gamma \leftrightarrow \gamma\varepsilon, \qquad \varepsilon\gamma\gamma \leftrightarrow \gamma\varepsilon,$$

$$M \leftrightarrow N.$$

Theorem 2. *Whatever the words P and Q in the alphabet B, $\mathfrak{K} : P \parallel Q$ if and only if $\mathfrak{M} : \psi(P)\gamma\varepsilon^4 \parallel \psi(Q)\gamma\varepsilon^4$.*

Denote by τ the following normal algorithm in the alphabet $B \cup \Delta$:

$$\beta \rightarrow \sigma\alpha,$$

$$\gamma \rightarrow \sigma,$$

$$\varepsilon \rightarrow \alpha\alpha.$$

Denote by $\mathfrak{N}$ the associative calculus in the alphabet B whose defining system is the system

$$\alpha\alpha\sigma\alpha\sigma \leftrightarrow \sigma\alpha\alpha, \qquad \alpha\alpha\sigma\sigma \leftrightarrow \sigma\alpha\alpha,$$

$$\tau(M) \leftrightarrow \tau(N).$$

Theorem 3. *Whatever the words P and Q in the alphabet B, in order that $\mathfrak{K} : P \parallel Q$, it is necessary and sufficient that $\mathfrak{N} : \tau(\psi(P)\gamma\varepsilon^4) \parallel \tau(\psi(Q)\gamma\varepsilon^4)$.*

Necessity follows immediately from Theorem 2 and the following easily verified assertions:

$$\mathfrak{N} : \tau(\varepsilon\beta\beta) \parallel \tau(\beta\varepsilon), \qquad \mathfrak{N} : \tau(\varepsilon\gamma\beta) \parallel \tau(\beta\varepsilon),$$

$$\mathfrak{N} : \tau(\varepsilon\beta\gamma) \parallel \tau(\gamma\varepsilon), \qquad \mathfrak{N} : \tau(\varepsilon\gamma\gamma) \parallel \tau(\gamma\varepsilon),$$

$$\mathfrak{N} : \tau(M) \parallel \tau(N).$$

From Lemma 2 and Theorem 3 it follows immediately that *for the calculus* $\mathfrak{N}$ *the problem of equivalence to a certain fixed word is undecidable.*

Remark. We have reduced the equivalence problem for the calculus $\mathfrak{C}_0$ to the equivalence problem for the calculus $\mathfrak{N}$. In an analogous way, *the equivalence problem for an arbitrary associative calculus can be reduced to the equivalence problem for a certain associative calculus in a two-letter alphabet, whose defining system consists of 3 relations.*

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CITED LITERATURE

[1] A. A. Markov, DAN, 55, No. 7, 587 (1947). [2] E. L. Post, J. Symb. Logic, 12, 1 (1947). [3] G. S. Tseitin, Tr. Matem. inst. im. V. A. Steklova AN SSSR, 52, 172 (1958). [4] D. Scott, J. Symb. Logic, 21, 111 (1956).

* 16 is the length of the word C_j.

** 32 is the length of the word D_j.

Note: Figure translations are in progress. See original paper for figures.