A DYNAMIC GAME OF PURSUIT

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Abstract

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A DYNAMIC GAME OF PURSUIT

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The pursuit takes place in the n-dimensional Euclidean space R^n. Two particles —the pursuer P and the evader E—with masses m_P and m_E, at the initial moment of time are located at the points q^0, r^0 and have momenta p^0 and s^0, respectively. The players P and E move in R^n, having the ability at each instant of time to change the direction of the applied forces, the magnitudes of which, generally speaking, may depend on the positions of the players and on their momenta. On R^{4n} there is given a certain smooth real function $U(z)$ (in what follows we shall use the notation $z = (q, p, r, s)$). Let $z(T)$ be the state of the system at the time $t = T$; then the objective of P is to maximize the quantity $U[z(T)]$, while player E pursues the opposite objective. At each instant of time the players have complete information about the state of the system.

Let us give a formal definition of the game. Let $\mathfrak{P}$ and $\mathfrak{E}$ be sets of vector functions

$$\varphi(z, T) = [\varphi_1(z, T), \dots, \varphi_n(z, T)],$$

$$\psi(z, T) = [\psi_1(z, T), \dots, \psi_n(z, T)],$$

having the following properties:

$$1. \quad \sum_{i=1}^{n} [\varphi_i(z, T)]^2 = \Phi^2(z).$$

$$2. \quad \sum_{i=1}^{n} [\psi_i(z, T)]^2 = \Psi^2(z).$$

$\Phi(z) \geq 0$, $\Psi(z) \geq 0$ are certain prescribed functions.

3. For any $\varphi \in \mathfrak{P}$ and $\psi \in \mathfrak{E}$ the system of equations

$$\dot{q}_i = \frac{p_i}{m_P}, \qquad \dot{p}_i = \varphi_i(z,T), \qquad \dot{r}_i = \frac{s_i}{m_E}, \qquad \dot{s}_i = \psi_i(z,T), \qquad i = 1, \dots, n, \tag{*}$$

has a unique solution for any initial conditions z^0.

The sets $\mathfrak{P}$ and $\mathfrak{E}$ are the sets of strategies of players P and E. In the situation (φ, ψ) the payoff function is defined as follows. Let $z(t)$ be the solution of the structural equations $(*)$ of the game in the situation (φ, ψ) under the initial conditions z^0. Then $K(z^0; \varphi, \psi) = U[z(T_0)]$, where T_0 is a prescribed in advance duration of the game, and $U(z)$ is a certain smooth real function given on R^{4n}.

Having specified the sets of strategies of the players and the payoff function in each situation, we have defined a certain pursuit game in normal form. We shall denote it by $\Gamma(z^0, T_0)$.

The following theorems hold (see ([1])).

Theorem 1. *In order that the game $\Gamma(z, T)$ have a continuously differentiable value in pure strategies, it is necessary that the problem*

for the Cauchy equation

$$\frac{\partial V}{\partial T} - \sum_{i=1}^{n} \left(\frac{\partial V}{\partial q_i} \frac{p_i}{m_P} + \frac{\partial V}{\partial r_i} \frac{s_i}{m_E} \right) + \Psi \left[\sum_{i=1}^{n} \left(\frac{\partial V}{\partial s_i} \right)^2 \right]^{1/2} - \Phi \left[\sum_{i=1}^{n} \left(\frac{\partial V}{\partial p_i} \right)^2 \right]^{1/2} = 0 \tag{1}$$

with the initial condition $V(z,T)\big|_{T=0} = U(z)$ has a solution.

Theorem 2. Let $V(z,T)$ be the solution of the Cauchy problem for equation (1), and let $\varphi^* = (\varphi_1^*, \dots, \varphi_n^*)$, $\psi^* = (\psi_1^*, \dots, \psi_n^*)$, where

$$\varphi_i^* = \Phi \frac{\partial V}{\partial p_i} \left[\sum_{i=1}^{n} \left(\frac{\partial V}{\partial p_i} \right)^2 \right]^{-1/2}, \qquad \psi_i^* = -\Psi \frac{\partial V}{\partial s_i} \left[\sum_{i=1}^{n} \left(\frac{\partial V}{\partial s_i} \right)^2 \right]^{-1/2} \tag{2}$$

belong to the sets $\mathfrak{P}$ and $\mathfrak{E}$, respectively; then

$$\operatorname{val} \Gamma(z,T) = V(z,T),$$

(where $\operatorname{val} \Gamma(z,T)$ denotes the value of the game), and the optimal strategies are determined by formulas (2).

In what follows we shall assume

$$\Phi = \text{const}, \qquad \Psi = \text{const}\,.$$

Put $n = 2$ and

$$U(z) = -\left[(q_1 - r_1)^2 + (q_2 - r_2)^2\right], \tag{3}$$

which corresponds to the case of pure pursuit in the plane.

By direct substitution we verify that the solution of the Cauchy problem for equation (1) with initial condition (3) has the form

$$\begin{aligned} V(q_1, q_2, p_1, p_2, r_1, r_2, s_1, s_2, T) = \; & -\left\{\left[\left(\frac{p_1}{m_P} - \frac{s_1}{m_E}\right) T + (q_1 - r_1)\right]^2 \right. \\ & \left. + \left[\left(\frac{p_2}{m_P} - \frac{s_2}{m_E}\right) T + (q_2 - r_2)\right]^2 \right\}^{1/2} \\ & - \left(\frac{\Phi}{m_P} - \frac{\Psi}{m_E}\right) \frac{T^2}{2} \Bigg\}^2 . \end{aligned} \tag{4}$$

We shall assume that:

$$1. \quad \frac{\Phi}{m_P} - \frac{\Psi}{m_E} > 0.$$

$$2. \quad \left[\sum_{i=1}^{2} \left[\left(\frac{p_i^0}{m_P} - \frac{s_i^0}{m_E}\right) T_0 + (q_i^0 - r_i^0)\right]^2\right]^{1/2} > \left(\frac{\Phi}{m_P} + \frac{\Psi}{m_E}\right) \frac{T_0^2}{2}.$$

It follows from condition 1 that the maneuverability of the pursuer P exceeds the maneuverability of the pursued E. Condition 2 means that capture cannot occur in the time $T \le T_0$.

Denote

$$\frac{\Phi}{m_P} - \frac{\Psi}{m_E} = h,$$

$$\left[\sum_{i=1}^{2} \left[\left(\frac{p_i^0}{m_P} - \frac{s_i^0}{m_E}\right) T_0 + (q_i^0 - r_i^0)\right]^2\right]^{1/2} = a^2.$$

According to Theorem 2, the optimal strategies of the players P and E are determined by the formulas (the optimal strategies could also be obtained directly, using the maximum principle (2))

$$\varphi_i^* = -\Phi \frac{(p_i/m_P - s_i/m_E)T + (q_i - r_i)}{\left[\sum_{i=1}^{2} \left[(p_i/m_P - s_i/m_E)T + (q_i - r_i)\right]^2\right]^{1/2}}, \tag{5}$$

$$\psi_i^* = -\Psi \frac{(p_i/m_P - s_i/m_E)T + (q_i - r_i)}{\left[\sum_{i=1}^{2} \left[(p_i/m_P - s_i/m_E)T + (q_i - r_i)\right]^2\right]^{-1/2}}, \qquad i = 1, 2. \tag{6}$$

Conditions 1 and 2 guarantee that the radical expression in (5), (6) is nonzero.

Optimal trajectories are determined from the system of equations

$$\ddot{x}_i = -\frac{h[\dot{x}_i(T_0 - t) + x_i]}{\left[\sum_{i=1}^{2} |\dot{x}_i(T_0 - t) + x_i|^2\right]^{1/2}}, \tag{7}$$

$$\ddot{c}_i = 0, \tag{8}$$

where $x_i = q_i - r_i,\ i = 1, 2$, are the coordinates of the relative radius-vector of the points P and E, and

$$c_i = \frac{1}{h}\left(\frac{\Phi}{m_P} r_i - \frac{\Psi}{m_E} q_i\right), \qquad i = 1, 2.$$

We shall call the point $c = (c_1, c_2)$ the **center of pursuit**. From (8) it follows that, in the course of the game in the equilibrium situation,

$$c_i(t) = A_i t + B_i, \qquad i = 1, 2, \tag{9}$$

where the constants A_i and B_i are determined from the initial conditions. The system of equations (7) can be reduced to a system of first-order equations

$$\dot{x}_1(T_0 - t) + x_1 = z\cos\theta,$$

$$\dot{x}_2(T_0 - t) + x_2 = z\sin\theta,$$

$$z\dot{\theta} = 0,$$

$$\dot{z} = -h(T_0 - t).$$

The optimal trajectory is determined from the equation

$$x_i(t) = -\frac{h}{2a^2}(x_i^0 + \dot{x}_i^0 T_0)t^2 + \dot{x}_i^0 t + x_i^0, \qquad i = 1, 2, \tag{10}$$

and equation (9), where $x_i^0, \dot{x}_i^0,\ i = 1, 2$, denote the corresponding quantities at the moment of time $t = 0$.

Let us now consider a pursuit game that differs from the game $\Gamma(z, T)$ in that the payoff function has the form

$$F(u) = F[-(q_1 - z_1)^2 - (q_2 - r_2)^2], \tag{11}$$

where $F(u)$ is some strictly monotonically increasing and differentiable function. Then, as is easy to verify, the solution of the Cauchy problem for equation (1) under the initial condition

$$V_1(z_1, T)\big|_{T=0} = F[-(q_1 - r_1)^2 - (q_2 - r_2)^2]$$

is equal to $F(V)$, where V is determined by formula (4). The optimal strategies have the same form as before and are determined by formulas (5), (6). The independence of the optimal strategies from the form of the function $F(U)$ is in complete agreement with known results of game theory (see ([3])).

In the case where the pursuit takes place in an n-dimensional Euclidean space R^n and the payoff function is defined as

$$U(q, p, r, s) = -\sum_{i=1}^{n}(q_i - r_i)^2,$$

the value function of the game is equal to

$$V(z, T) = -\left\{ \left[\sum_{i=1}^{n} \left[\left(\frac{p_i}{m_P} - \frac{s_i}{m_E} \right) T + (q_i - r_i) \right]^2 \right]^{1/2} - \frac{hT^2}{2} \right\}^2 .$$

In conclusion, we note that if a solution of the Cauchy problem for equation (1), satisfying the requirements of Theorem 2, exists, then it is unique among solutions of this type. This follows from Theorem 2 and from the uniqueness of the value function of the game.

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Note: Figure translations are in progress. See original paper for figures.