ON THE CHOICE OF A PARAMETER IN SOLVING FUNCTIONAL EQUATIONS BY THE REGULARIZATION METHOD

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.948.35

MATHEMATICS

V. A. MOROZOV

ON THE CHOICE OF A PARAMETER IN SOLVING FUNCTIONAL EQUATIONS BY THE REGULARIZATION METHOD

(Presented by Academician Yu. N. Rabotnov on 28 X 1966)

1. Let H be a real Hilbert space, F a linear normed space, and A some linear bounded operator defined on H and mapping H into F. We consider the problem

$$Au = \bar{f}, \qquad \bar{f} \in F \tag{1}$$

of finding the function $\bar{u} = R[\bar{f}; A] \in H$ satisfying equation (1). Suppose that the function $\bar{f} \in A[H] \subset F$; the function $\bar{u}$ is determined uniquely by the datum $\bar{f}$. The collection $\{\bar{f}; A\}$ will be called the **initial data** of problem (1).

Let $S(A_h) = \{A_h,\ 0 < h \leq h_0\}$ be a family of linear bounded operators A_h, defined on H, such that $\|A - A_h\|_{H \to F} \leq \zeta(h) \to 0$ as $h \to 0$. Any collection $\{f_\delta; A_h\}$, $f_\delta \in F$, $\|f_\delta - \bar{f}\|_F < \delta$, $A_h \in S(A_h)$, will be called **approximate data** of problem (1). We consider the problem of stable solution of equation (1), i.e., determination from the approximate data $\{f_\delta; A_h\}$ of a function $u_{\delta h} \in H$ such that $\|\bar{u} - u_{\delta h}\| \to 0$ as $\delta, h \to 0$, where $\| \cdot \|$ denotes the norm in H. Similar problems were considered in works ([1–5]).

Suppose that it is known a priori that $\bar{u} \in D$, where D is some weakly closed convex set of the space H (the case $D = H$ is allowed). Define the parametric functional ([1])

$$\Phi^\alpha_{\delta h}[u] = \Phi^\alpha[u; f_\delta, A_h; u^*] = \|A_{hu} - f_\delta\|^2_F + \alpha\|u - u^*\|^2, \qquad u \in D, \tag{2}$$

where $\alpha > 0$ is a parameter, and $u^* \in D$ is a prescribed function.

Theorem 1. There exists a unique function $u^\alpha_{\delta h} \in D$ satisfying the relation

$$\Phi^\alpha_{\delta h}[u^\alpha_{\delta h}] = \inf_{u \in D} \Phi^\alpha_{\delta h}[u]. \tag{3}$$

The proof is based on the inequality

$$\alpha\|(u_1-u_2)/2\| \leq \tfrac{1}{2}\Phi^{\alpha}_{\delta h}[u_1]+\tfrac{1}{2}\Phi^{\alpha}_{\delta h}[u_2]-\Phi^{\alpha}_{\delta h}[(u_1+u_2)/2], \qquad u_1,u_2\in H,$$

and is carried out analogously to work ([3]).

Define the functions

$$\varphi_{\delta h}(\alpha)=\Phi^{\alpha}_{\delta h}[u^{\alpha}_{\delta h}], \qquad \gamma_{\delta h}(\alpha)=\|u^{\alpha}_{\delta h}-u^{*}\|^2,$$

$$\rho_{\delta h}(\alpha)=\|A^{\alpha}_{hu_{\delta}h}-f_{\delta}\|^2_F.$$

Lemma 1. For every $\alpha>0$ the function $\varphi_{\delta h}(\alpha)$ is continuous, monotonically increasing, and

$$\lim_{\alpha\to+\infty}\varphi_{\delta h}(\alpha)=\|A^{*}_{hu}-f_{\delta}\|^2_F.$$

Lemma 2. For any $\alpha>0$ the function $\gamma_{\delta h}(\alpha)$ is continuous and monotonically decreasing; moreover,

$$\lim_{\alpha\to+\infty}\gamma_{\delta h}(\alpha)=0.$$

Lemma 3. For any $\alpha>0$ the function $\rho_{\delta h}(\alpha)$ is continuous, monotonically increasing, and takes values in the interval

$$(\overline{\zeta}(h;\bar{u})+\delta^2,\ \|A^{*}_{hu}-f_{\delta}\|^2_F], \qquad \overline{\zeta}(h;\bar{u})=\|A_h\bar{u}-A\bar{u}\|_F.$$

Remark 1. If the range $Q_{A_h}=A_h(H)\subseteq F$ of each operator in the family $S(A_h)$ is everywhere dense in F, then one may assert that the function $\rho_{\delta h}(\alpha)$ takes values in the interval $(0,\|A^{*}_{hu}-f_{\delta}\|^2_F$. This is fulfilled, for example, if $A_h\equiv E$ and the space H is embedded in F ([6]), i.e., for the problem of reconstructing functions ([5]).

From the stated assertions it follows easily:

Theorem 2 (discrepancy principle). Let, for any δ and h, $0<\delta\leq\delta_0$, $0<h\leq h_0$, the condition

$$\kappa(\delta,h;\bar{u},u^{*})=(\overline{\zeta}(h;\bar{u})+\delta)(1+\beta(\delta,h))^{1/2}<\|A^{*}_{hu}-f_{\delta}\|_F, \tag{4}$$

be satisfied, where $\beta=\beta(\delta,h)$, $\beta(0,0)=0$, is some positive bounded function continuous at the point $(0,0)$.

Then there exists at least one value of the regularization parameter $\alpha = \alpha(\delta, h) \geq 0$ such that

$$\rho_{\delta h}(\alpha(\delta, h)) = \kappa^2(\delta, h; \bar{u}, u^*), \tag{5}$$

and in this case

$$\lim_{\delta,h\to 0} \|u_{\delta h} - \bar{u}\| = 0.$$

Remark 2. If Q_{A_h} is everywhere dense in F, then in relation (4) one may set $\beta(\delta, h) \equiv 0$.

Remark 3. If the space F is Hilbert, then for sufficiently small δ and h the value of the parameter $\alpha = \alpha(\delta, h)$ is determined uniquely by condition (5).

Remark 4. If $\zeta(h; \bar{u}) \leq \delta$, then the discrepancy principle in fact does not depend on the sought solution $\bar{u} = R[\bar{f}; A]$ and is quite effective.

We note that if problem (1) has a nonunique solution, then by $\bar{u} = R[\bar{f}; A]$ one may understand the generalized normal solution ([7]) of this problem, which is uniquely determined by the formula

$$\bar{u} = \mathrm{pr}_{N_A} u^* + \mathrm{pr}_{H_A} \tilde{u}, \qquad \tilde{u} = R[\bar{f}; A],$$

where $N_A = \{u;\ Au = 0\}$, $H_A = H \dot{-} N_A$.

The formulated theorem is proved analogously to Theorem 4 ([5]).

2. The following result is a refinement of Theorem 3 ([5]) and can serve as a basis for choosing suitable values of the regularization parameter, as well as for estimating the deviation of the regularized solution from the exact one, if some a priori information about the exact solution of problem (1) is available. Namely, the following is true.

Theorem 3. Let F be a unitary space, the element u^α realize the lower bound of the functional $\Phi^\alpha[u; \bar{f}, A; u^*]$, $u \in H$, and the element u^α_δ of the functional $\Phi^\alpha[u; f_\delta, A; u^*]$, $u \in H$. Then

$$\|u^\alpha_\delta - \bar{u}\| \leq \omega(\alpha; \bar{u}, u^*) + \delta/\sqrt{\alpha},$$

where $\omega(\alpha; \bar{u}, u^*) = \|u^\alpha - \bar{u}\| \to 0$ as $\alpha \to 0$. If $u^* - \bar{u} = A^* v$, where $v \in F$, then $\omega(\alpha; \bar{u}, u^*) \leq \sqrt{\alpha}\|v\|_F$, and if $u^* - \bar{u} = A^* A \bar{v}$, $\bar{v} \in H$, then $\omega(\alpha; \bar{u}, u^*) \leq \alpha\|\bar{v}\|$.

3. Let the function $\bar{u} = \bar{u}(x) \in W_2^{(1)}[a, b]$ ([6])* be the (unique) solution of the integral equation

$$K[u] \equiv A(x)u(x) + \int_a^b k(x,\xi)u(\xi)\,d\xi = \bar{f}(x), \qquad a \leq x, \xi \leq b, \tag{6}$$

where $A = A(x)$, $k = k(x,\xi)$, and $\bar{f} = \bar{f}(x)$ are continuous functions of their arguments. Suppose that continuous functions $\tilde{f}(x)$ and $\delta(x) \geq 0$ are known and are related to $\bar{f}(x)$ by

$$|\bar{f}(x) - \tilde{f}(x)| \leq \delta(x). \tag{7}$$

Using some ideas of paper ([9]) and of the regularization method, we define the generalized approximate solution of equation (6) with approximate data $\{\tilde{f}, K\}$ as the solution $\tilde{u}(x)$ of the extremal problem

$$\|\tilde{u} - u^*\|^2_{W_2^{(1)}} = \inf_{u\in\Omega} \|u - u^*\|^2_{W_2^{(1)}}, \tag{8}$$

where the set Ω is defined as follows:

$$\Omega = \left\{ u \in W_2^{(1)};\ \left| A(x)u(x) + \int_a^b k(x,\xi)u(\xi)\,d\xi - \tilde{f}(x) \right| \leq \delta(x) \right\}.$$

It is obvious that the set Ω is closed, convex, and nonempty ($\bar{u} \in \Omega$ by virtue of (7)).

Theorem 4. There exists a unique function $\tilde{u} \in W_2^{(1)}$ realizing equality (8), i.e., the generalized approximate solution of equation (6) is uniquely determined by prescribing the functions $\tilde{f}(x)$ and $\delta(x)$ satisfying relation (7).

Theorem 5. Let $\varepsilon > 0$ be given. There exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that, if the relation $\|\delta\|_{L_2} \leq \delta_0$ is satisfied, the generalized approximate solution of equation (6) will belong to the ε-neighborhood of the function $\bar{u} = \bar{u}(x)$: $\|\tilde{u} - \bar{u}\|_{W_2^{(1)}} < \varepsilon$.

Remark 5. If $k(x,\xi) \equiv \delta(x-\xi)$, then we arrive at the problem of reconstructing a function $\bar{u} \in W_2^{(1)}$. The function $\tilde{f}(x)$ may be interpreted as an experimentally obtained function representing $\bar{u} = \bar{u}(x)$ with some small error in L_2. Equality (8) in this case expresses definite requirements on the process of processing experimental information. The validity of Theorems 4 and 5 is preserved.

4. Let us write the extremal problem (8) in discrete form. Suppose that on the interval $[a,b]$ two grids of nodes ($n \geq m$) are defined: $\{x\}$: $a \leq x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots < x_n \leq b$, $\{\xi\}$: $a \leq \xi_1 < \xi_2 < \cdots < \xi_j < \xi_{j+1} < \cdots < \xi_m \leq b$.

Let us write the approximate equality

$$\int_a^b k(x,\xi)u(\xi)\,d\xi \simeq \sum_{j=1}^{m} \chi_j k(x,\xi_j)u_j,$$

where $u_j = u(\xi_j)$; χ_j are the coefficients of some quadrature formula, and define the set

$$\Omega_h = \left\{ u_j,\ j = 1,2,\dots,m;\ \left| \sum_{j=1}^{m} \chi_j k_{ij} u_j - \tilde{f}_i \right| \le \Delta_i,\ 1 \le i \le n \right\},$$

where $k_{ij} = k(x_i,\xi_j)$, $\tilde{f}_i = \tilde{f}(x_i)$, and $\Delta_i \ge \delta_i$ are chosen so that $\bar{u}_j = \bar{u}(\xi_j) \in \Omega_h$.

* The norm in the space $W_2^{(1)}$ is defined by the relation $\|\bar{u}\|_{W_2^{(1)}} = \|\bar{u}'\|^2_{L_2} + \|\sqrt{q}\,\bar{u}\|^2_{L_2}$, where $q = q(x) > 0$ is a continuous function.

Then we arrive at the following problem:

Find the quantities $\tilde{u}_j \in \Omega_h$ for which the condition

$$\Phi[\tilde{u}_1,\tilde{u}_2,\dots,\tilde{u}_m] = \inf_{u_j\in\Omega_h} \Phi[u_1,u_2,\dots,u_m],$$

is satisfied,

$$\Phi[u_1,u_2,\dots,u_m] = \sum_{j=1}^{m-1} \left\{ \left(\frac{u_{j+1}-u_j}{\xi_{j+1}-\xi_j} \right)^2 + \frac{q_{j+1}u_{j+1}^2 + q_j u_j^2}{2} \right\} (\xi_{j+1}-\xi_j),$$

i.e., to the problem of quadratic programming ([10], [11]). The presence of a priori constraints on the desired solution can in most cases be reflected in the definition of the set of admissible grid functions Ω_h.

Moscow State University
named after M. V. Lomonosov

Received
19 X 1966

REFERENCES

1. A. N. Tikhonov, DAN, 151, No. 3, 501 (1963); 153, No. 1, 49 (1963).

2. I. N. Dombrovskaya, V. K. Ivanov, *Siberian Mathematical Journal*, 6, No. 3, 499 (1965).

3. V. A. Morozov, *Vestn. Mosk. Univ.*, Ser. Math. and Mech., No. 4, 13 (1965).

4. V. A. Morozov, *Zhurnal Vychislitel' noi Matematiki i Matematicheskoi Fiziki*, 6, No. 1, 170 (1966).

5. V. A. Morozov, DAN, 167, No. 3, 508 (1966).

6. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.

7. A. N. Tikhonov, DAN, 163, No. 3, 591 (1965).

8. S. G. Mikhlin, *The Problem of the Minimum of a Quadratic Functional*, M.-L., 1952.

9. L. V. Kantorovich, *Siberian Mathematical Journal*, 3, No. 5, 701 (1962).

10. G. P. Kuntsi, V. Krepl, *Nonlinear Programming*, M., 1965.

11. G. Zoutendijk, *Methods of Feasible Directions*, IL, 1963.

Note: Figure translations are in progress. See original paper for figures.