ON THE INDICATOR OF FUNCTIONS OF INTEGER ORDER, ANALYTIC AND OF COMPLETELY REGULAR GROWTH IN A HALF-PLANE

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Abstract

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ON THE INDICATOR OF FUNCTIONS OF INTEGER ORDER, ANALYTIC AND OF COMPLETELY REGULAR GROWTH IN A HALF-PLANE

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B. Ya. Levin and A. Pfluger showed ([1], Chs. I, III; [2,3]) that an entire function of order $\rho > 0$ has completely regular growth if and only if the set of its zeros $\{z_n\}$ is regularly distributed, i.e.:

1. For all $\theta_k \in [0, 2\pi] \setminus N$, where N is at most countable, there exists the limit

$$\lim_{r\to\infty} \frac{n(r, \theta_1, \theta_2)}{r^\rho} = \Delta(\theta_1, \theta_2), \qquad (1)$$

called the angular density (see [1], p. 118).

2. In the case of integer ρ, the zeros z_n are arranged with a special symmetry, namely, there exists the limit

$$\lim_{r\to\infty} \sum_{|z_n|<r} z_n^{-\rho} = \sigma \neq \infty. \qquad (2)$$

In the works of the authors cited, a formula for the indicator of these entire functions was also obtained. In article [4] these results were extended to functions of noninteger order, regular in a half-plane. For them, the concept of argument density, which is an analogue of angular density, is introduced in a special way, and a formula for the indicator is obtained. In the present note we consider functions of integer order in a half-plane. Definitions of the order of a function and of functions of completely regular growth (in an open or closed angle) are given in article [4].

Theorem 1. *Every function, regular and of integer order $\rho \geq 0$ in $\operatorname{Im} z > 0$, is uniquely representable in the form*

$$f(z) = \exp\left\{ i\sum_{k=0}^{\rho} a_k z^k + \frac{1}{\pi i}\left[\int_{-1}^{1} \frac{\ln|f(t)|}{t-z}\,dt + \int_{-1}^{1} \frac{i\,d\varphi(t)}{t-z} + z^{\rho+1}\int_{|t|>1} \frac{\ln|f(t)|\,dt}{t^{\rho+1}(t-z)} + z^{\rho+1}\int_{|t|>1} \frac{d\varphi(t)}{t^{\rho+1}(t-z)}\right]\right\} \prod_{|z_n|\le 1} \frac{z-z_n}{z-\bar{z}_n} \times$$

$$\times \prod_{|z_n|>1} \frac{\left(1-\dfrac{z}{z_n}\right)\exp\left[\dfrac{z}{z_n} + \dfrac{1}{2}\left(\dfrac{z}{z_n}\right)^2 + \dots + \dfrac{1}{\rho}\left(\dfrac{z}{z_n}\right)^\rho\right]}{\left(1-\dfrac{z}{\bar{z}_n}\right)\exp\left[\dfrac{z}{\bar{z}_n} + \dfrac{1}{2}\left(\dfrac{z}{\bar{z}_n}\right)^2 + \dots + \dfrac{1}{\rho}\left(\dfrac{z}{\bar{z}_n}\right)^\rho\right]}, \qquad (3)$$

where a_k are real constants; z_n are the interior (lying in $\operatorname{Im} z > 0$) zeros of $f(z)$; $\varphi(t)$ is a real nondecreasing function for which $\varphi'(t) = 0$ almost everywhere and the integral

$$\int_{|t|\ge 1} \frac{d\varphi(t)}{t^{\rho+2}}$$

converges.

The theorem is readily obtained from the analogous theorem given in ([4]).*

* For the case of integer order, Theorem 1 is in a certain sense more convenient than the one mentioned ($\varphi(t)$ and a_k in these theorems are different).

Starting from representation (3), for an arbitrary function $f(z)$, regular and of integer order $\rho > 0$ in $\operatorname{Im} z > 0$, we introduce the following notation:

$$\tau(t) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_1^t \frac{\ln|f(x)|}{x}\,dx + \frac{1}{2\pi}\int_1^t \frac{d\varphi(x)}{x}, & t > 1, \\ \dfrac{1}{2\pi}\displaystyle\int_t^{-1} \frac{\ln|f(x)|}{|x|}\,dx + \frac{1}{2\pi}\int_t^{-1} \frac{d\varphi(x)}{|x|}, & t < -1, \end{cases} \qquad (4)$$

$$c(r, \eta_1, \eta_2) = \sum_{\eta_1 < \theta_n \le \eta_2,\ 1 \le r_n \le r} \sin\theta_n, \qquad r > 1 \quad (z_n = r_n e^{i\theta_n}), \qquad (5)$$

$$a(r, \eta_1, \eta_2) = \begin{cases} c(r, \eta_1, \eta_2), & 0 < \eta_1 < \eta_2 < \pi, \\ c(r, 0, \eta_2) - \tau(r), & 0 = \eta_1 < \eta_2 < \pi, \\ c(r, \eta_1, \pi) - \tau(-r), & 0 < \eta_1 < \eta_2 = \pi, \\ c(r, 0, \pi) - \tau(r) - \tau(-r), & \eta_1 = 0,\ \eta_2 = \pi, \end{cases} \qquad (6)$$

$$a(r,\eta_1,\eta_2) = -a(r,\eta_2,\eta_1), \quad \eta_1 > \eta_2; \qquad a(r,\eta,\eta) \equiv 0.$$

Definition 1. If for a function $f(z)$, regular and of order $\rho > 0$ in $\operatorname{Im} z > 0$, for all $\eta_1, \eta_2 \in [0 \le \theta \le \pi] \setminus N$, where N is at most countable and contains no points $\eta = 0$, $\eta = \pi$, there exists the finite limit

$$\lim_{r\to\infty} \frac{a(r,\eta_1,\eta_2)}{r^\rho} = \lambda(\eta_1,\eta_2),$$

then we shall say that the set of zeros of the function $f(z)$ has an **argument-boundary density** in the half-plane $\operatorname{Im} z > 0$.

Definition 2. A function $f(z)$ of order $\rho > 0$ in $\operatorname{Im} z > 0$ is called a **function of finite type** if, asymptotically,

$$\sup_{|z|\le r,\ \operatorname{Im} z>0} |f(z)| < \exp(Kr^\rho), \qquad K = \text{const}. \tag{7}$$

Denote by A_ρ (by $\overline{A}_\rho$) the class of functions regular of order ρ and of finite type in $\operatorname{Im} z > 0$, having completely regular growth in the open angle $0 < \arg z < \pi$ (in the closed angle $0 \le \arg z \le \pi$).

Theorem 2. Let the function $f(z)$ be regular of integer order $\rho > 0$ and of finite type in $\operatorname{Im} z > 0$. Then, in order that $f(z)$ belong to the class A_ρ, it is necessary and sufficient that the set of its zeros have an argument-boundary density and that there exist the finite limit

$$\sigma = \lim_{r\to\infty}\left[\frac{1}{\rho}\sum_{1\le r_n\le r}\frac{\sin\rho\theta_n}{r_n^\rho} - \frac{1}{2\pi}\int_{1\le|x|\le r}\frac{\ln|f(x)|}{x^{\rho+1}}\,dx - \frac{1}{2\pi}\int_{1\le|x|\le r}\frac{d\varphi(x)}{x^{\rho+1}}\right]. \tag{8}$$

Theorem 3. If $f(z) \in A_\rho$ and $\rho > 0$ is an integer, then the indicator of the function $f(z)$ is expressed by the formula

$$h_f(\theta) = (2\sigma - a_\rho)\sin\rho\theta + 2\cos\rho\theta\int_0^\pi \psi\,\frac{\sin\rho\psi}{\sin\psi}\,d\lambda(\psi) +$$

$$+2\int_0^\theta \frac{\psi}{\sin\psi}\sin\rho(\theta-\psi)\,d\lambda(\psi) + 2\int_\theta^\pi \frac{\psi-\pi}{\sin\psi}\sin\rho(\theta-\psi)\,d\lambda(\psi), \tag{9}$$

where $\lambda(\psi) \equiv \lambda(0,\psi)$ is the argument-boundary density of the zeros of $f(z)$, and σ and a_ρ are defined respectively by relations (8) and (3).

Remark. For writing formula (9) in other forms, Theorem 4 may be useful.

Theorem 4. If $\lambda(\psi) = \lambda(0, \psi)$ is the argument-boundary density of the function $f(z) \in A_\rho$, $\rho > 0$ an integer, then the equality

$$\int_0^\pi \frac{\sin \rho\psi}{\sin \psi}\, d\lambda(\psi) = 0 \tag{10}$$

holds.

Definition 3. Let the function $f(z)$ be regular and of order $\rho > 0$ in $\operatorname{Im} z > 0$, and let $\tau(r)$ be defined by equality (4). Then the limits

$$\lim_{r\to+\infty} \frac{\tau(r)}{r^\rho} = l_1, \qquad \lim_{r\to+\infty} \frac{\tau(-r)}{r^\rho} = l_2, \tag{11}$$

if they exist and are finite, are called, respectively, the right- and left-hand boundary density of the set of zeros of the function $f(z)$. If in (11) upper limits are taken, then we shall call them upper boundary densities (l_1^* and l_2^*).

Definition 4. Let a set of points $\{z_n\}$, $n = 1, 2, ...$, be given in $\operatorname{Im} z > 0$, all of whose limit points lie on the real axis. Then, if for all $\eta_1, \eta_2 \in (0 \le \eta \le \pi) \setminus N$, where N is at most countable and does not contain $\eta = 0, \eta = \pi$, there exists the finite limit

$$\lim_{r\to\infty} \frac{c(r, \eta_1, \eta_2)}{r^\rho} = \mu(\eta_1, \eta_2), \tag{12}$$

then we shall say that the set $\{z_n\}$ has, in the domain $\operatorname{Im} z > 0$, an argument density with exponent ρ (or an upper argument density $\mu^*(\eta_1, \eta_2)$, if in (12) an upper limit is taken).

Theorem 5. If $f(z)$ is a function of order $\rho \ge 1$ and of finite type in $\operatorname{Im} z > 0$, then its upper boundary densities are finite, and its upper argument density is bounded.

Theorem 6. If $f(z)$ is a function of class A_ρ and $\rho > 0$ is an integer, then the indicator of $f(z)$ can be expressed by the formula $(0 < \theta < \pi)$

$$h_f(\theta) = 2\Bigg\{ \sin \rho\theta \cdot \left[\sigma - l_1^* + (-1)^\rho l_2^* - \frac{a_\rho}{2}\right] + \cos \rho\theta \cdot \left[\int_0^\pi \psi \frac{\sin \rho\psi}{\sin \psi}\, d\mu^*(\psi) + \pi\rho(-1)^\rho l_2^*\right]$$
$$+ \int_0^\theta \frac{\psi}{\sin \psi} \sin \rho(\theta - \psi)\, d\mu^*(\psi) + \int_\theta^\pi \frac{\psi - \pi}{\sin \psi} \sin \rho(\theta - \psi)\, d\mu^*(\psi) \Bigg\}.$$

Theorem 7. Let $f(z)$ be regular in $\operatorname{Im} z > 0$ and continuous in $\operatorname{Im} z \ge 0$, and have integer order $\rho > 0$ and finite type. Then $f(z) \in A_\rho$ if and only if:

1. The zeros of $f(z)$ have boundary and argument density, the latter being continuous at $\psi = 0$ and $\psi = \pi$.
2. The function $\varphi(t)$, defined from (3), satisfies the equality

$$\lim_{r\to\infty} \frac{1}{r^\rho} \int_{1\le|t|\le r} \frac{d\varphi(t)}{|t|} = 0. \tag{13}$$

3. There exists the finite limit (8).
4. The indicator $h_f(\theta)$ is continuous at $\theta = 0$ and $\theta = \pi$.

The indicator of the function $f(z) \in \overline{A}_\rho$ has the following form $(0 \le \theta \le \pi)$:

$$h_f(\theta) = 2\Bigg\{ \sin\rho\theta \cdot \left[\sigma - l_1 + (-1)^\rho l_2 - \frac{a_\rho}{2}\right] + \cos\rho\theta \cdot \left[\int_0^\pi \psi \frac{\sin\rho\psi}{\sin\psi}\, d\mu(\psi) + \pi\rho(-1)^\rho l_2\right] + \int_0^\theta \frac{\psi}{\sin\psi} \sin\rho(\theta-\psi)\, d\mu(\psi) + \int_\theta^\pi \frac{\psi-\pi}{\sin\psi} \sin\rho(\theta-\psi)\, d\mu(\psi)\Bigg\}.$$

Theorem 8. If the assumptions of Theorem 7 are satisfied and ρ is odd, then $f(z) \in \overline{A}_\rho$ if and only if:

1. The set of zeros of the function $f(z)$ has, in the open angle $0 < \theta < \pi$, an angular density (see (1), where it is assumed that $0 < \theta_k < \pi$).
2. The limit (8) exists.
3. The indicator $h_f(\theta)$ satisfies the condition

$$h_f(0) + h_f(\pi) = 2\pi \int_{+0}^{\pi-0} \sin\rho\psi\, d\Delta(\psi).$$

We note that the last improper integral certainly converges for every function of the class A_ρ (([4], p. 498)).

In conclusion we give two theorems concerning functions of finite degree, i.e., functions of order not exceeding one and of finite type.

Definition 5. A function analytic in $\operatorname{Im} z \geqslant 0$, whose zeros $z_n = r_n e^{i\theta_n}$ satisfy the condition (see ([1], p. 289))

$$\sum_{n=1}^{\infty} \frac{\sin\theta_n}{r_n} < \infty,$$

is called a function of the class A.

Theorem 9. In order that a function $f(z)$, regular and of finite degree in $\operatorname{Im} z \geqslant 0$, belong simultaneously to the classes A and A_1, it is necessary and sufficient that the limits

$$\lim_{r\to\infty}\frac{1}{r}\int_1^r \frac{\ln|f(-x)|}{x}\,dx = l_2, \qquad \lim_{r\to\infty}\frac{1}{r}\int_1^r \frac{\ln|f(x)|}{x}\,dx = l_1$$

(the left and right boundary densities) exist, and that the integral

$$\int_1^\infty \frac{\ln|f(x)f(-x)|}{x^2}\,dx \tag{14}$$

converge.

Theorem 10. In order that a function, regular and of finite degree in $\operatorname{Im} z \geqslant 0$, belong simultaneously to the classes A and $\overline{A}_1$, it is necessary and sufficient that the integral (14) converge and that the condition

$$h_f(0) + h_f(\pi) = 0.$$

be satisfied.

It must be noted that, for the case of sufficiency, Theorem 10 was established earlier (see ([1], p. 317)).

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Note: Figure translations are in progress. See original paper for figures.