ON A MAXIMUM PRINCIPLE FOR HYPERBOLIC EQUATIONS AND ITS APPLICATION TO EQUATIONS OF MIXED TYPE

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Abstract

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MATHEMATICS

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ON A MAXIMUM PRINCIPLE FOR HYPERBOLIC EQUATIONS AND ITS APPLICATION TO EQUATIONS OF MIXED TYPE

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§ 1. Consider the equation

$$\mathcal{L}[u] \equiv u_{\xi\eta} + a(\xi;\eta)u_\xi + b(\xi;\eta)u_\eta + c(\xi;\eta)u = 0 \qquad (\mathcal{L})$$

in the characteristic triangle $O_0A_0C_0$ (the domain Δ, $O_0(0;0)$, $A_0(\xi_0;-\xi_0)$, $C_0(0;-\xi_0)$, $\xi_0>0$), $a, a_\xi, b, c \in C^{(0)}(\bar{\Delta} \setminus \overline{O_0A_0})$. Let

$$h = a_\xi + ab - c = h_1 + h_2, \qquad h_1 \leq 0 \text{ in } \Delta; \quad \beta = \exp\left\{\int_0^\xi b\, d\xi\right\}; \quad \alpha = \beta a;$$

$$\gamma = -\beta h_1 - \beta h_2 = \gamma_1 + \gamma_2; \qquad c = c_1 + c_2, \qquad c_1 \leq 0 \text{ in } \Delta.$$

We shall say that the coefficients of $(\mathcal{L})$ satisfy in Δ conditions (C) if: 1) $a<0$ on $\overline{O_0C_0} \setminus O_0$; 2)

$$-a(P) > \int_{PQ} [2|\gamma_2| + \beta|c_2|]\, d\xi$$

in $\Delta \cup C_0A_0$, and conditions (D), if in $\Delta \cup C_0A_0$: 1) $a<0$, 2)

$$-\beta(Q)a(Q) > \int_{PQ} |\gamma|\, d\xi,$$

where $P \in \overline{O_0C_0} \setminus O_0$, PQ is a segment of the characteristic $\eta = \text{const}$. We shall call a function $u(\xi;\eta)$ a solution of equation $(\mathcal{L})$ of class $[R]$ if $\mathcal{L}[u] \equiv 0$ in Δ, $u \in C^{(2)}(\Delta)$, $u \in C^{(0)}(\bar{\Delta})$, $u \in C^{(1)}(\bar{\Delta} \setminus \overline{O_0A_0})$, $u|_{\overline{O_0C_0}} = 0$,

$$\max_{\bar{\Delta}} u > 0, \qquad \max_{\bar{\Delta}} u \geq -\min_{\bar{\Delta}} u.$$

Theorem 1 (maximum principle). *Let the coefficients of $(\mathcal{L})$ satisfy in Δ conditions (C) or (D), and let $u(\xi;\eta)$ be an arbitrary solution of $(\mathcal{L})$ of class $[R]$.*

Then $\max_{\bar{\Delta}} u$ is attained only on $\overline{O_0A_0}$.

Suppose the contrary. Let

$$\max_{\bar{\Delta}} u = u(Q), \qquad Q \in \bar{\Delta} \setminus \overline{O_0A_0}.$$

Represent $(\mathcal{L})$ in the form [(1)]

$$(\beta u_\eta)_\xi + (\alpha u)_\xi + \gamma u = 0$$

and integrate this equation along the segment PQ:

$$\beta(Q)u_\eta(Q) = \left\{-a(P)u(Q) + \int_{PQ} [u(Q) - u]\gamma_2 \, d\xi - u(Q)\int_{PQ} \beta c_2 \, d\xi\right\} + \int_{PQ} [u(Q) - u]\gamma_1 \, d\xi - u(Q)\int_{PQ} \beta c_1 \, d\xi; \tag{1}$$

$$\beta(Q)u_\eta(Q) = -\beta(Q)a(Q)u(Q) - \int_{PQ} \gamma u \, d\xi. \tag{2}$$

By virtue of conditions (C) and (D), respectively, from (1) and (2) we obtain $u_\eta(Q) > 0$, which is impossible and proves the theorem.

Example 1. Denote by $(\mathcal{L}_m^{\lambda\mu})$ the equation $(\mathcal{L})$ in which

$$a = \frac{\sigma_1}{2(\xi+\eta)} + \frac{f(\xi;\eta)}{(-\xi-\eta)^{2\sigma_1+\lambda}}, \qquad b = \frac{\sigma_1}{2(\xi+\eta)} + \frac{g(\xi;\eta)}{(-\xi-\eta)^{2\sigma_1+\lambda}},$$

$$c = \frac{r(\xi;\eta)}{(-\xi-\eta)^{2\sigma_1+\mu}};$$

$$0 < m < 2, \qquad \lambda < \frac{2-m}{2+m}, \qquad \mu < \frac{4-m}{2(2+m)}, \qquad \sigma_1 = \frac{m}{2+m},$$

$$f,\ (\xi+\eta)f_\xi,\ g,\ r \in C^{(0)}(\overline{\Delta}).$$

Lemma 1. *If the length of the segment O_0A_0 is sufficiently small, then the maximum principle is valid for the equation $(\mathcal{L}_m^{\lambda\mu})$.*

Indeed, for a sufficiently small length of O_0A_0 the coefficients of $(\mathcal{L}_m^{\lambda\mu})$ satisfy conditions (C)

$$a(P) = \frac{\sigma_1}{2\eta}\left[1 - (-\eta)^{\sigma}\frac{2}{\sigma_1}f(0;\eta)\right], \qquad \sigma = \frac{2-m}{2+m} - \lambda > 0,$$

$$h = -\frac{m(4+m)}{4(2+m)}\frac{1}{(\xi+\eta)^2}\left[1 - (-\xi-\eta)^{\sigma}\mathfrak{W}(\xi;\eta)\right],$$

$$-a(P) - \int_{PQ}\beta|c|\,d\xi = -\frac{\sigma_1}{2\eta}\left[1 - (-\eta)^{\sigma}\Omega(\xi;\eta)\right],$$

$$\mathfrak{W}(\xi;\eta) = \frac{4(2+m)^2}{m(4+m)}\left[\left(\lambda + \frac{3}{2}\sigma_1\right)f(\xi;\eta) - \frac{\sigma_1}{2}g(\xi;\eta) - (\xi+\eta)f_{\xi}(\xi;\eta) + \right.$$

$$\left. + (-\xi-\eta)^{\sigma}f(\xi;\eta)g(\xi;\eta) - (-\xi-\eta)^{1+\lambda-\mu}r(\xi;\eta)\right],$$

$$\Omega(\xi;\eta) = \frac{2}{\sigma_1}\left[f(0;\eta) + (-\eta)^{3/2\,\sigma_1+\lambda}\omega(\xi;\eta)\right],$$

$$\omega(\xi;\eta) = \int_0^{\xi}\frac{|r(t;\eta)|}{(-t-\eta)^{3/2\,\sigma_1+\mu}}\exp\left\{\int_0^t\frac{g(\tau;\eta)}{(-\tau-\eta)^{2\sigma_1+\lambda}}\,d\tau\right\}dt.$$

Example 2.

$$\mathfrak{M}_{\lambda\mu}[u] \equiv u_{\xi\eta} + \left(\frac{\lambda}{\xi+\eta} - \frac{\mu}{\xi-\eta}\right)u_{\xi} + \left(\frac{\lambda}{\xi+\eta} + \frac{\mu}{\xi-\eta}\right)u_{\eta} = 0. \qquad (\mathfrak{M}_{\lambda\mu})$$

Lemma 2. *For the equation $(\mathfrak{M}_{\lambda\mu})$, for any $0 \le \lambda \le 1$, $0 \le \mu \le 1$, $\lambda + \mu \ne 0$, the maximum principle is valid.*

Indeed, in Δ conditions (C) are satisfied

$$a(P) = \frac{\lambda+\mu}{\eta} < 0; \qquad h_1 = -\frac{\lambda-\lambda^2}{(\xi+\eta)^2} \le 0; \qquad h_2 = \frac{\mu-\mu^2}{(\xi-\eta)^2} \ge 0,$$

$$-a(P) - 2\int_0^{\xi}|\gamma_2|\,d\xi > -\frac{\lambda+\mu}{\eta} + \frac{2\mu}{\eta}(1 - 2^{\mu-1}) > 0.$$

It can be shown that for the equation $(\mathfrak{M}_{\lambda\mu})$ with $\lambda = 0$, $0 < \mu < 1$ (the Euler-Poisson-Darboux equation) the well-known maximum principle for hyperbolic equations does not hold (([1]), theorem 1).

§ 2. Consider the general generalized F. Tricomi equation

$$T_m[u] \equiv \operatorname{sgn} y \cdot |y|^m u_{xx} + u_{yy} + \frac{M(x;y)}{|y|^\beta} u_x + \frac{N(x;y)}{|y|^\alpha} u_y + \frac{F(x;y)}{|y|^\gamma} u = 0 \qquad (T_m)$$

in the domain D, bounded by: 1) a simple Jordan arc Γ, situated in the upper half-plane and resting on the axis $y = 0$ at the points $O(0;0)$ and $A(a;0)$, $a > 0$; 2) the characteristics OC and AC. Let D_1 and D_2 be, respectively, the subdomains of ellipticity and hyperbolicity of the equation (T_m); OA is the line of transition; $M, N, F \in C^{(0)}(\overline{D_1})$; $M, N, F \in C^{(0)}(\overline{D_2})$; $F \leq 0$ in D_1; $M, N \in C^{(1)}(\overline{D_2})$; $N \geq 0$ in D_1 only when $\alpha > 0$; $m \geq 0$, $\beta < 1 - m/2$, $\alpha < 1$, $\gamma < 1 - m/4$. We shall call the function $u(x;y)$ a **regular solution** of the equation $T_m[u] \equiv G(x;y)$ $(G(x;y) \in C^{(0)}(\overline{D}))$, if $T_m[u] \equiv G(x;y)$ in $D_1 \cup D_2$; $u \in C^{(2)}(D_1 \cup D_2)$; $u \in C^{(0)}(\overline{D})$; $u \in C^{(1)}[D_1 \cup (D_2 \setminus \overline{OA})]$; $u_y(x;0+0) = u_y(x;0-0)$ for $0 < x < a$, and these limits are finite–the gluing condition. A regular solution of the equation (T_m) will be regarded as belonging to the class $[P_0]$ if

$$u\big|_{\overline{OC}} \equiv 0, \qquad \max_{\overline{D}} u > 0, \qquad \max_{\overline{D}} u \geq -\min_{\overline{D}} u.$$

Problem T (F. Tricomi). Find a regular solution $u(x,y)$ of the equation $T_m[u] = G(x;y)$ from the data: 1) $u|_\Gamma = \varphi$, 2) $u|_{\overline{OC}} = \psi$, where φ and ψ are continuous functions; $\varphi(0) = \psi(0)$.

We shall say that for equation (T_m) the principle of the (strict) maximum for problem T is valid if an arbitrary solution of class $[P_0]$ attains its maximum in $\overline{D}$ (only) in the closure of the "elliptic" arc Γ.

Theorem 2. *If in equation* (T_m), $0 < m < 2$, $\alpha = \beta = \gamma = 0$, *and the length of the transition line is sufficiently small, then the principle of the strict maximum for problem T is valid for it.*

Suppose the contrary. Let $u(x;y) \in [P_0]$, $\max_{\overline{D}} u = u(Q)$, but $Q \in \overline{D} \setminus \overline{\Gamma}$. By the known property of elliptic equations and Lemma 1 the point $Q \in OA$. Consequently, by the modified lemma of E. Hopf ([1]), Lemma 2, $u_y(x_Q;0+0) < 0$. But $u_y(x_Q;0-0) \geq 0$, which contradicts the gluing condition and proves what is required.

Theorem 3. *If in equation* (T_m), $0 < m < 2$, *and the length of the transition line is sufficiently small, then the maximum principle for problem T is valid for it.*

Suppose the contrary. Let $u(x;y) \in [P_0]$, $\max_{D_1} u = u(Q)$, but $Q \in \overline{\Gamma}$. Consequently, $Q \in OA$. But then, by virtue of the lemma of K. I. Babenko ([2]) (one can show that its assertion is also valid in the case $\alpha > 0$, but with $N \geq 0$ in D_1), $u_y(x_Q; 0+0) < 0$, which is impossible by the gluing condition.

Theorem 4. *Let the coefficients of an equation of the form ($\mathcal{L}$), corresponding to the equation (T_m), satisfy in the domain Δ the conditions (C) or (D). Then for equation (T_m) the following are valid: 1) the principle of the strict maximum for problem T, if $\alpha = \beta = \gamma = 0$; 2) the maximum principle for problem T.*

The proof is analogous to the proofs of Theorems 2 and 3.

Theorem 5. *For the equation $T_m[u] = G(x;y)$, the solution of problem T is unique if $0 < m < 2$ and the length of the transition line is sufficiently small.*

The proof follows easily from Theorems 2 and 3.

Theorem 6. *If the coefficients of equation (T_m) satisfy the conditions of Theorem 4, then for the equation $T_m[u] = G(x;y)$ the solution of problem T is unique.*

§ 3. Consider an equation of mixed type with two mutually perpendicular parabolicity lines of the first kind

$$T_{mn}[u] \equiv \operatorname{sgn} y \cdot |y|^m U_{xx} + \operatorname{sgn} x \cdot |x|^n u_{yy} = 0, \qquad (T_{mn})$$

$$m \geq 0, \qquad n \geq 0, \qquad m + n \neq 0$$

in a domain D, consisting of three subdomains D_1, D_2, D_3. The domain $D_1(x > 0;\ y > 0)$ is bounded by the coordinate axes and by a simple Jordan arc Γ resting on them at the points $A(a;0)$ and $B(0;b)$. The domains $D_2(x > 0;\ y < 0)$ and $D_3(x < 0;\ y > 0)$ are bounded respectively by two pairs of characteristics OC and AC, OE and BE. We shall call a function $u(x;y)$ a regular solution of equation (T_{mn}) if $T_{mn}[u] \equiv 0$ in $D_1 \cup D_2 \cup D_3$, $u \in C^{(2)}(D_1 \cup D_2 \cup D_3)$, $u \in C^{(0)}(\overline{D})$, $u \in C^{(1)}[D_1 \cup (\overline{D}_2 \setminus \overline{OA}) \cup (\overline{D}_3 \setminus \overline{OB})]$,

$$u_y(x; 0+0) = u_y(x; 0-0) \quad \text{for } 0 < x < a, \qquad u_x(0+0; y) = u_x(0-0; y)$$

for $0 < y < b$, and these limits are finite—the gluing conditions. We shall regard a regular solution $u(x;y)$ of equation (T_{mn}) as belonging to the class $[P_{00}]$, if

$$u|_{\overline{OC} \cup \overline{OE}} \equiv 0, \qquad \max_{\overline{D}} u > 0, \qquad \max_{\overline{D}} u \geq -\min_{\overline{D}} u.$$

Problem T. Find a regular solution $u(x;y)$ of equation (T_{mn}) from the data: 1) $u|_\Gamma = \varphi$, 2) $u|_{\overline{OC}} = \psi_1$, 3) $u|_{\overline{OE}} = \psi_2$, where φ, ψ_1, ψ_2 are continuous functions; $\psi_1(0) = \psi_2(0)$.

Theorem 7 (the strong maximum principle for problem T). An arbitrary solution of the equation (T_{mn}) of class $[P_{00}]$ attains its maximum in $\bar{D}$ only in the closure of the "elliptic" arc Γ.

The proof is analogous to Theorem 2 and follows from the modified E. Hopf lemma [1] and Lemma 2.

Theorem 8. For the equation (T_{mn}), the solution of problem T is unique.

The proof follows from Theorem 7.

In conclusion we note that a somewhat different extremum principle for problem T in the case of the Lavrent' ev–Bitsadze and Tricomi equations belongs to A. V. Bitsadze [3].

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Note: Figure translations are in progress. See original paper for figures.