ON A THEOREM OF LYAPUNOV TYPE

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.9

MATHEMATICS

M. M. KHAPAEV

ON A THEOREM OF LYAPUNOV TYPE

(Presented by Academician A. N. Tikhonov, 30 XII 1966)

Consider the system of equations

$$dx_i/dt = f_i(t, x_1, \dots, x_n), \qquad 1 \le i \le n. \tag{1}$$

We shall assume that the right-hand sides f_i in the domain

$$|x_i| \le H, \qquad t \ge 0 \tag{2}$$

are continuous and satisfy the Lipschitz condition with constant N with respect to the variables x_i. Moreover, as is usual in such problems, $f_i(t, 0, \dots, 0) = 0$, i.e., the system (1) has the trivial solution (equilibrium point).

We shall also consider a system with constantly acting perturbations

$$dx_i/dt = f_i(t, x_1, \dots, x_n) + \mu R_i(t, x_1, \dots, x_n); \tag{3}$$

here μ is a parameter, and the functions R_i are defined in the domain (2) and satisfy conditions ensuring the existence of a continuous solution.

The stability under constantly acting perturbations of the trivial solution of the nonlinear system (1) was studied by the method of Lyapunov functions (^{1}), under the assumption that the conditions of Lyapunov' s theorem on asymptotic stability are satisfied, or else asymptotic stability of the equilibrium point was required ($^{2-4}$), etc. Under such restrictions, the result of (^{5}) was obtained, connected with averaging on an unbounded interval of systems that can be written in the form (3), where some of the f_i are identically equal to zero.

The present paper is devoted to the study of stability under constantly acting perturbations of the trivial solution of a nonlinear system of differential equations of the general form (1), under the assumption that only the conditions of Lyapunov' s theorem on stability are satisfied, i.e., in the "neutral" case. In this case stability is determined by the properties of the perturbations μR_i. In the conditions of the theorem formulated below, it is assumed that there exists a negative mean value of the scalar product of grad v (where v is the Lyapunov function) with the perturbation vector $\bar{R}$, the mean being computed along the integral curves of the unperturbed system (1). These conditions, together with certain smoothness requirements, are sufficient for stability under constantly acting perturbations.

Previously, stability in the "neutral" case was studied for systems admitting linearization (see, for example, (^{7})).

We formulate the main result of the present paper in the form of a theorem.

Theorem. *Let the following conditions be satisfied:*

a) *There exists a positive definite Lyapunov function*

$$v(t, x_1, \dots, x_n),$$

which admits an infinitesimal upper limit (^{1}).

b) *The total derivative of v, formed in accordance with equations (1), is nonpositive in the domain (2):*

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i \leq 0.$$

c) The partial derivatives $\partial v / \partial x_i$ are continuous with respect to x_i in the domain $|x_i| \leq H$, uniformly with respect to t on the interval $0 \leq t < \infty$.

d) The perturbations $R_i(t, x_1, \dots, x_n)$ in the domain $|x_i| \leq H$ are continuous with respect to x_i, uniformly with respect to t on the interval $0 \leq t < \infty$, and are bounded by the constant M.

e) Uniformly with respect to t_0 and x_{i0}, there exists the mean value

$$\varphi_0(x_{10}, \dots, x_{n0}) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} R_i(t, \bar{x}_1, \dots, \bar{x}_n)\, dt;$$

where the integral is computed along the solution $\bar{x}_i$ of system (1) with initial conditions x_{i0}; moreover, outside an arbitrarily small neighborhood of the equilibrium point the mean φ_0 is strictly less than 0:

$$\varphi_0(x_{10}, \dots, x_{n0}) < -\delta^2 < 0.$$

If these conditions are satisfied, then for any $\varepsilon > 0$ one can specify such $\eta(\varepsilon)$ and $\mu_0(\varepsilon)$ that any solution of system (3), $x_i(t)$, with initial values x_{i0} satisfying the inequalities $|x_{i0}| < \eta(\varepsilon)$, for $\mu < \mu_0(\varepsilon)$, satisfies the inequalities $|x_i(t)| < \varepsilon$ for all $t > 0$.

The proof of the theorem will consist in indicating a method by which, for any $\varepsilon > 0$, $\eta(\varepsilon)$ and $\mu_0(\varepsilon)$ are constructed.

Thus, let a number $\varepsilon > 0$ be given; also introduce the number ε_1 by the condition $0 < \varepsilon_1 < \varepsilon$. Conditions a) and b), according to Lyapunov' s theorem on stability (^{1}), determine the stability of the trivial solution of system (1), i.e., one can specify such an $\eta_1 < \varepsilon_1$ that solutions which, at the initial moment $t = t_0$, satisfy the condition $|\bar{x}_{i0}| < \eta_1$, will for all $t > t_0$ obey the inequalities $|\bar{x}_i(t)| < \varepsilon_1$.

According to condition a) of the theorem, there exists a positive definite function $w(x_1, \dots, x_n)$, independent of t, such that

$$v(t, x_1, \dots, x_n) \geq w(x_1, \dots, x_n). \tag{4}$$

Denote by w_{η_1} the exact lower bound of the function $w(x_1, \dots, x_n)$, if the set of all possible values x_i is connected by the condition $\max\{|x_1|, \dots, |x_n|\} = \eta_1$. The surface $w(x_1, \dots, x_n) = w_{\eta_1}$ lies in the domain $|x_i| < \eta_1$. The moving surface $v(t, x_1, \dots, x_n) = w_{\eta_1}$, by virtue of inequality (4), lies entirely inside the surface $w = w_{\eta_1}$ for all $t > 0$. Since v admits an arbitrarily small upper bound, there is a sufficiently small number $\eta > 0$ such that the η-neighborhood of the equilibrium point $|x_i| < \eta$ lies entirely inside the moving surface $v(t, x_1, \dots, x_n) = w_{\eta_1}$, for all $t > 0$.

From a point lying in the η-neighborhood of the equilibrium point, let us issue at the moment $t = 0$ an integral curve of system (3), $x_i(t)$. Suppose further that the solution $x_i(t)$ has left the domain $|x_i| < \eta$ and at some moment $t = t_0$ has crossed the surface $v(t, x_1, \dots, x_n) = w_{\eta_1}$. We shall consider the behavior of the function v along the solution $x_i(t)$; for this, we form the derivative of the function v by virtue of the equations of system (3):

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i + \mu \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} R_i = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i + \mu\varphi(x(t)), \tag{5}$$

where it is denoted that

$$\sum_{i=1}^{n} \frac{\partial v}{\partial x_i} R_i(t, x_1(t), \dots, x_n(t)) = \varphi(x(t)).$$

By condition (b) of the theorem, in the domain (2)

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i \leq 0.$$

We shall

estimate $\int_{t_0}^{t} \varphi(x(t))\,dt$, which enters into the estimate for v obtained by integrating (5):

$$v(t, x_1(t), \dots, x_n(t)) \le v_0(t_0, x_{10}, \dots, x_{n0}) + \mu \int_{t_0}^{t} \varphi(x(t))\,dt. \tag{6}$$

Adding and subtracting $\int_{t_0}^{t} \varphi(\tilde{x}(t))\,dt$, we obtain

$$\int_{t_0}^{t} \varphi(x(t))\,dt = \int_{t_0}^{t} \varphi(\tilde{x}(t))\,dt + \int_{t_0}^{t} [\varphi(x(t)) - \varphi(\tilde{x}(t))]\,dt. \tag{7}$$

Here $\tilde{x}(t)$ denotes the collection $\tilde{x}_i(t)$—the solution of system (1) issuing at the moment $t = t_0$ from the point x_{i0}.

Conditions a) and d) of the theorem make it possible to estimate $\int_{t_0}^{t} \varphi(\tilde{x}(t))\,dt$. In view of this condition there exists a mean value $\varphi_0(x_{10}, \dots, x_{n0}) < -\delta^2 < 0$, since the point x_{i0} lies outside the domain $|x_i| < \eta$. By the definition of the mean (8), there exists a function $\chi(t)$, whose limit as $t \to \infty$ is 0, such that

$$\int_{t_0}^{t} \varphi(\tilde{x}(t))\,dt = (t - t_0)[\varphi_0(x_{10}, \dots, x_{n0}) + \chi(t)]. \tag{8}$$

(We note that averaging along the integral curves of the degenerate system was proposed and applied in the works (^{5,6}).) We shall construct our estimates for the function v for $t_0 \le t \le t_0 + 2l$, where l is chosen so large that

$$|\chi(t)| < \delta^2/4 \quad \text{for } t > t_0 + l. \tag{9}$$

We estimate the integral of the difference in (7), choosing the parameter μ sufficiently small so that the solution $x_i(t)$ on the interval $[t_0, t_0 + 2l]$ does not leave the domain $|x_i| < \varepsilon$, and also so that, in all, $\int_{t_0}^{t} \varphi(x(t))\,dt$ is negative for $t_0 + l \le t \le t_0 + 2l$.

Let us estimate the difference $x_i(t) - \tilde{x}_i(t)$:

$$x_i(t) - \tilde{x}_i(t) = \int_{t_0}^{t} [f(t, x_1, \dots, x_n) - f(t, \tilde{x}_1, \dots, \tilde{x}_n)]\,dt + \mu \int_{t_0}^{t} R_i(t, x_1, \dots, x_n)\,dt.$$

Using the Lipschitz inequality for f_i and the boundedness of R_i in the domain (2), we obtain, for $t_0 \leq t \leq t_0 + 2l$,

$$|x_i - \tilde{x}_i| \leq N \int_{t_0}^{t} |x_i - \tilde{x}_i| \, dt + \mu M 2l,$$

whence, according to Gronwall' s lemma, we obtain

$$|x_i - \tilde{x}_i| < \mu M \cdot 2l e^{2Nl}. \tag{10}$$

The solution $\tilde{x}_i(t)$ lies in the domain $|x_i| < \varepsilon_1$; therefore, in order that the solution $x_i(t)$ not leave the domain $|x_i| < \varepsilon$ for $t_0 \leq t \leq t_0 + 2l$, it is sufficient to require that $|x_i - \tilde{x}_i| < \varepsilon - \varepsilon_1$; for this we choose

$$\mu < \mu_1 = \frac{\varepsilon - \varepsilon_1}{M \cdot 2l e^{2Nl}}.$$

Assumptions c) and d) of the theorem ensure the continuity of $\varphi(x(t))$ in the domain (2); therefore one can choose such a μ_2 that, for all $\mu < \mu_2$, by virtue of the continuity of $\varphi(x(t))$ and inequality (10),

$$\int_{t_0}^{t} |\varphi(x(t)) - \varphi(\tilde{x}(t))| \, dt < (t - t_0) \frac{\delta^2}{4}. \tag{11}$$

Choose now $\mu_0 = \min\{\mu_1, \mu_2\}$; then for $\mu < \mu_0$, for $t_0 \leq t \leq t_0 + 2l$, the condition $|x_i(t)| < \varepsilon$ will be satisfied and the estimate (11) will hold. Inequalities (9) and (11), relations (7) and (8), and also condition d) of the theorem make it possible to estimate $\int_{t_0}^{t} \varphi(x(t)) \, dt$ for $t_0 + l < t < t_0 + 2l$:

$$\int_{t_0}^{t} \varphi(x(t)) \, dt < (t - t_0) \left[\varphi_0(x_{10}, \dots, x_{n0}) + \chi(t) + \delta^2/4\right] <$$

$$< (t - t_0) \left[-\delta^2 + \delta^2/4 + \delta^2/4\right] = (t - t_0)(-\delta^2/2). \tag{12}$$

Thus, the integral $\int_{t_0}^{t} \varphi(x(t)) \, dt$, at least starting with $t = t_0 + l$, becomes negative. After leaving the surface $v(t, x_1, \dots, x_n) = w_{\eta_1}$, the integral curve $x_i(t)$, by virtue of the choice $\mu < \mu_0$, remains in the domain $|x_i(t)| < \varepsilon$ for $t_0 \leq t \leq t_0 + 2l$.

In inequality (6) the second term becomes negative and, consequently, along the solution $x_i(t)$ the Lyapunov function $v(t, x_1, \dots, x_n)$ begins to decrease, so that a moment t_1, $t_0 < t_1 < t_0 + 2l$, will occur when the curve returns inside the surface $v(t, x_1, \dots, x_n) = w_{\eta_1}$; moreover, $v(t_0, x_{10}, \dots, x_{n0}) = v(t_1, x_{11}, \dots, x_{n1}) = w_{\eta_1}$. All estimates are uniform with respect to t_0; therefore the solution may leave the surface $v(t, x_1, \dots, x_n) = w_{\eta_1}$ and return to it arbitrarily many times, and for all $t > 0$ it remains in the domain $|x_i(t)| < \varepsilon$. The theorem is proved.

Remark. Conditions c) and d) of the theorem can easily be replaced by less restrictive ones. Suppose that there exist a summable function $M(t)$ and a constant M_0 such that, for $t \geq 0$ and $|x_i| \leq H$,

$$|R_i(t, x_1, \dots, x_n)| < M(t),$$

and on any finite interval $[t_1, t_2]$

$$\int_{t_1}^{t_2} M(t)\, dt \leq M_0(t_2 - t_1).$$

This assumption makes it possible to obtain the estimate (10) without using the boundedness of R_i.

In obtaining estimate (11), the continuity of the product

$$\frac{\partial v}{\partial x_i} R_i(t, x_1, \dots, x_n)$$

is used, which is also not necessary. To estimate the difference $\varphi(x(t)) - \varphi(\tilde{x}(t))$ for small $x - \tilde{x}$, we shall assume that there exist a summable function $F(t)$ and a constant F_0, as well as a nondecreasing function $\psi(\alpha)$, $\lim_{\alpha \to 0} \psi(\alpha) = 0$, such that, for $t \geq 0$ and $|x_i| \leq H$,

$$|\varphi(x') - \varphi(x'')| < \psi(|x' - x''|) F(t); \qquad \int_{t_1}^{t_2} F(t)\, dt \leq F_0(t_2 - t_1)$$

on any finite interval $[t_1, t_2]$.

Moscow State University
named after M. V. Lomonosov

Received
28 XII 1966

CITED LITERATURE

1. A. M. **Lyapunov**, *The General Problem of the Stability of Motion*, Kharkov, 1892; 3rd ed., Moscow, 1950.

2. I. G. **Malkin**, *Theory of Stability of Motion*, 2nd ed., "Nauka," 1966.

3. L. **Massera**, Ann. Math., 50, No. 3 (1949).

4. V. E. **Germanadze**, N. N. **Krasovskii**, PMM, 21, No. 6, 769 (1957).

5. V. M. **Volosov**, DAN, 145, No. 5 (1962).

6. V. M. **Volosov**, DAN, 141, No. 1 (1961).

7. A. M. **Molchanov**, DAN, 141, No. 1 (1961).

8. N. N. **Bogolyubov**, Yu. A. **Mitropolsky**, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.