# ON SELF-ADJOINT EXTENSIONS OF CERTAIN SINGULAR PARTIAL DIFFERENTIAL OPERATORS

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## Abstract

## Full Text

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*MATHEMATICS*

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# ON SELF-ADJOINT EXTENSIONS OF CERTAIN SINGULAR PARTIAL DIFFERENTIAL OPERATORS

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One of the methods of proving existence theorems for elliptic formally adjoint operators in bounded domains is the method based on self-adjoint extensions (see, for example, ($^1$)). In the present article a class of singular partial differential operators is indicated for which existence theorems can also be obtained by the method of self-adjoint extensions.

Consider, in the $(n+1)$-dimensional Euclidean space of points $z = (x, y)$ $(x = (x_1, \dots, x_n))$, a domain $\Omega^+$ situated in the half-space $y > 0$ and adjacent to the hyperplane $y = 0$. The boundary $\partial\Omega^+$ of the domain $\Omega^+$ is divided into a part $\Gamma^0$, lying in the hyperplane $y = 0$, and a part $\Gamma^+$, lying in the half-space $y > 0$. Let $C_0^\infty(\Omega^+)$ denote the set of functions each of which is infinitely differentiable and has compact support contained in $\Omega^+$.

On the indicated set of functions we introduce for consideration a linear differential operator of order $2m$, having the form

$$\begin{aligned} \mathcal{L}u = (-1)^m \sum_{|\alpha|=m} C_m^{(\alpha)} D_B^\alpha \left( \sum_{|\beta|=m} C_m^{(\beta)} a_{\alpha\beta}(z) D_B^\beta u(z) \right) + \\ + (-1)^m \sum_{|\bar\alpha|=m} C_m^{(\bar\alpha)} D_B^\alpha \left[ \frac{\partial}{\partial y} \left( \sum_{|\bar\beta|=m} C_m^{(\bar\beta)} a_{\bar\alpha\bar\beta}(z) \frac{\partial}{\partial y} D_B^\beta u(z) \right) \right. \\ \left. + \sum_{|\bar\beta|=m} C_m^{(\bar\beta)} \frac{k}{y} a_{\bar\alpha\bar\beta}(z) \frac{\partial}{\partial y} D_B^\beta u(z) \right]. \end{aligned} \tag{1}$$

Here we put

$$\alpha = (\alpha', 2\alpha_{n+1}), \qquad \alpha' = (\alpha_1, \dots, \alpha_n),$$

$$|\alpha| = |\alpha'| + 2\alpha_{n+1} = \alpha_1 + \dots + \alpha_n + 2\alpha_{n+1}, \qquad D_B^{\alpha} = D_B^{\alpha'} = D_x^{\alpha'} B_y^{2\alpha_{n+1}}, \quad (2)$$

$$D_x^{\alpha'} = \partial^{|\alpha'|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}, \qquad \bar{\alpha} = (\alpha', 2\alpha_{n+1}+1), \qquad |\bar{\alpha}| = \alpha_1+\dots+\alpha_n+2\alpha_{n+1}+1,$$

$$C_m^{(\alpha)} = \frac{m!}{\alpha_1! \cdots \alpha_n! (2\alpha_{n+1})!}, \qquad C_m^{(\bar{\alpha})} = \frac{m!}{\alpha_1! \cdots \alpha_n! (2\alpha_{n+1}+1)!}.$$

An analogous meaning is assigned to other symbols similar to them. The symbol $B_y$, as usual, denotes the Bessel operator

$$\frac{\partial^2}{\partial y^2} + \frac{k}{y}\frac{\partial}{\partial y} \quad (k > 0,\ y \geq 0).$$

As regards the coefficients $a_{\alpha\beta}(z)$ and $a_{\bar{\alpha}\bar{\beta}}(z)$, it is assumed that they are bounded and are continuously differentiable a sufficient number of times in $\overline{\Omega^+}$, in the sense of applying the corresponding number of times the operators occurring in (1).

In addition, it is assumed that they are real and symmetric. The real homogeneous form

$$a(z,\xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(z) C_m^{(\alpha)} C_m^{(\beta)} \xi^{\alpha}\xi^{\beta} + \sum_{|\bar{\alpha}|=|\bar{\beta}|=m} a_{\bar{\alpha}\bar{\beta}}(z) C_m^{(\bar{\alpha})} C_m^{(\bar{\beta})} \xi^{\bar{\alpha}}\xi^{\bar{\beta}} \quad (3)$$

of the real variables $\xi_1, \dots, \xi_n, \xi_{n+1}$ is positive definite.

Moreover, it is assumed that for any position of the point $z$ in the domain $\Omega^+$ the inequality

$$a(z,\xi) \geq \mu \left( \sum_{|\alpha|=m} C_m^{(\alpha)} |\xi^{\alpha}|^2 + \sum_{|\bar{\alpha}|=m} C_m^{(\bar{\alpha})} |\xi^{\bar{\alpha}}|^2 \right) = \mu \sum_{|\gamma|=m} C_m^{(\gamma)} |\xi^{\gamma}|^2, \quad (4)$$

holds, where $\mu$ is a positive constant,

$$\xi = (\xi_1, \dots, \xi_n, \xi_{n+1}), \qquad \xi^{\alpha} = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \xi_{n+1}^{2\alpha_{n+1}}, \qquad C_m^{(\gamma)} = \frac{m!}{\gamma_1! \cdots \gamma_n! \gamma_{n+1}!},$$

$$\xi^{\bar{\alpha}} = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \xi_{n+1}^{2\alpha_{n+1}+1}.$$

Condition (4) is called the condition of $B$-ellipticity of the operator $\mathcal{L}$ (2). We shall consider the operator $\mathcal{L}$ as an operator acting in the space $\mathcal{L}_{2,k}(\Omega^+)$. Here $\mathcal{L}_{2,k}(\Omega^+)$ denotes the space of square-summable functions with weight $y^k$ ($k > 0$) in the domain $\Omega^+$. Form the scalar product

$$(\mathcal{L}u, u)_k = \int_{\Omega^+} u\,\mathcal{L}u\,y^k\,dz. \tag{5}$$

Integrating $m$ times by parts and taking into account that $u \in C_0^\infty(\Omega^+)$, we obtain

$$(\mathcal{L}u, u)_k = \int_{\Omega^+} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(z) C_m^{(\alpha)} C_m^{(\beta)} D_B^\alpha u\, D_B^\beta u\, y^k\,dz +$$

$$+ \int_{\Omega^+} \sum_{|\bar\alpha|=|\bar\beta|=m} a_{\bar\alpha\bar\beta}(z) C_m^{(\bar\alpha)} C_m^{(\bar\beta)} \frac{\partial}{\partial y} D_B^\alpha u\, \frac{\partial}{\partial y} D_B^\beta u\, y^k\,dz. \tag{6}$$

Hence, by virtue of inequality (4), we have

$$(\mathcal{L}u, u)_k \geq \mu \left( \int_{\Omega^+} \sum_{|\alpha|=m} C_m^{(\alpha)} |D_B^\alpha u|^2 y^k\,dz + \int_{\Omega^+} \sum_{|\bar\alpha|=m} C_m^{(\bar\alpha)} \left| \frac{\partial}{\partial y} D_B^\alpha u \right|^2 y^k\,dz \right). \tag{7}$$

Put

$$r^2 = \sum_{i=1}^n x_i^2 + y^2,$$

and let $\Delta_B$ denote the singular Beltrami operator

$$\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + B_y.$$

In [3] it was proved that the fundamental solution of the operator $\Delta_B^m$ with singularity at the origin has the form

$$u(x, y) = \begin{cases} C_1 r^{2m-\gamma} \ln r, & \text{if } 2m \geq \gamma \text{ and } \gamma \text{ is even,} \\ C_2 r^{2m-\gamma}, & \text{in all other cases,} \end{cases} \tag{8}$$

where $\gamma = n + k + 1$.

To obtain a fundamental solution with singularity at an arbitrary point, one must apply to the function $u(x, y)$ the generalized shift operator $T_{s,t}^{x,y}$ (see [3]). Then the fundamental solution will have the form

$$E(x, y; s, t) = C_3 \int_0^\pi \left[ \sum_{i=1}^n (x_i - s_i)^2 + y^2 + t^2 - 2yt \cos\alpha \right]^{(2m-\gamma)/2} \times$$

$$\times \ln \left[ \sum_{i=1}^n (x_i - s_i)^2 + y^2 + t^2 - 2yt \cos\alpha \right]^{1/2} \sin^{k-1} \alpha \, d\alpha,$$

if $2m \geq \gamma$ and $\gamma$ is an even number;

$$E(x, y; s, t) = C_4 \int_0^\pi \left[ \sum_{i=1}^n (x_i - s_i)^2 + y^2 + t^2 - 2yt \cos\alpha \right]^{\frac{2m-\gamma}{2}} \sin^{k-1} \alpha \, d\alpha \quad (9)$$

in all other cases.

It is proved that for any $u \in C_0^\infty(\Omega^+)$ the integral representation

$$u(x, y) = \frac{1}{C_{m,n}^{(k)}} \int_{\Omega^+} \sum_{|\alpha|=m} C_m^{(\alpha)} D_B^\alpha u D_B^\alpha E t^k \, ds \, dt +$$

$$+ \frac{1}{C_{m,n}^{(k)}} \int_{\Omega^+} \sum_{|\bar{\alpha}|=m} C_m^{(\bar{\alpha})} \frac{\partial}{\partial t} D_B^\alpha u \frac{\partial}{\partial t} D_B^\alpha E t^k \, ds \, dt. \qquad (10)$$

Here $s = (s_1, \ldots, s_n)$, $t \geqslant 0$, and $(s, t)$ is a point of $(n+1)$-dimensional Euclidean space.

Estimates of the fundamental solution and its derivatives, when the singularity of the fundamental solution lies inside the domain (see (4)), show that the kernels of the integral operators standing on the right in representation (10) are either bounded or have a weak singularity. If, however, the point $z$ approaches the part of the boundary $\Gamma^0$, then the singularity of the kernels increases. The presence of the weight $t^k$ in the integral representation (10) compensates for the excessive singularity arising in this case. But then from representation (10) we find that

$$\|u\|^2_{\mathcal{L}_{2,k}(\Omega^+)} \leqslant C \sum_{|\alpha|=m} C_m^{(\alpha)} \int_{\Omega^+} |D_B^\alpha u|^2 t^k \, ds \, dt +$$

$$+ C \sum_{|\bar{\alpha}|=m} C_m^{(\bar{\alpha})} \int_{\Omega^+} \left| \frac{\partial}{\partial t} D_B^\alpha u \right|^2 t^k \, ds \, dt. \qquad (11)$$

The latter inequality, together with inequality (7), shows that

$$(\mathcal{L}u, u)_k \geqslant \frac{\mu}{C}\|u\|_k^2. \tag{12}$$

It is not hard to verify that the operator $\mathcal{L}$ is a symmetric operator. Inequality (12) shows that it is also positive definite.

Let now $\mathcal{L}$ be a formally self-adjoint operator of $B$-elliptic type such that its restriction $\mathcal{L}_0(D(\mathcal{L}_0) = C_0^\infty(\Omega^+))$ is a positive definite operator, i.e. $(\mathcal{L}_0 u, u)_k \geqslant \mu(u, u)_k$ for any $u \in C_0^\infty(\Omega^+)$ with a positive constant $\mu$ independent of $u$. We now consider the self-adjoint positive definite Friedrichs extension of the operator $\mathcal{L}_0$: $(\widetilde{\mathcal{L}})^* = \widetilde{\mathcal{L}} \supset \mathcal{L}_0$.

It is known that the range of the operator $\widetilde{\mathcal{L}}$ coincides with the whole space (in our case with $\mathcal{L}_{2,k}(\Omega^+)$). Consequently, the equation

$$\widetilde{\mathcal{L}}u = f \tag{13}$$

is uniquely solvable. As for functions from the domain of the operator $\widetilde{\mathcal{L}}$, i.e. from the set $D(\widetilde{\mathcal{L}})$, we shall say that they satisfy the generalized Dirichlet condition, without specifying here for the time being whether this should mean the fulfillment of boundary conditions on the entire boundary $\partial\Omega^+$, or on a part of it. The solution of equation (13) is an element of the space $\mathcal{L}_{2,k}(\Omega^+)$. Above we showed that the restriction $\mathcal{L}_0$ of the operator $\mathcal{L}$ to infinitely differentiable functions with compact supports lying in $\Omega^+$ is positive definite.

Let $H_{\mathcal{L}_0}$ be the completion of the set $D(\mathcal{L}_0) = C_0^\infty(\Omega^+)$ in the metric $\|u\|_{\mathcal{L}}^2 = (\mathcal{L}_0 u, u)_k$. Then the domain of definition of the Friedrichs extension $\widetilde{\mathcal{L}}$ of the operator $\mathcal{L}_0$ is described by the relation

$$D(\widetilde{\mathcal{L}}) = D(\mathcal{L}_0^*) \cap H_{\mathcal{L}_0}. \tag{14}$$

Apparently, the norm $\|u\|_{\mathcal{L}}$ will be equivalent to the norm

$$\|u\|_1 = \int_{\Omega^+} \sum_{|\alpha|=m} |D_B^\alpha u|^2 y^k \, dz + \int_{\Omega^+} \sum_{|\bar{\alpha}|=m} \left|\frac{\partial}{\partial y} D_B^\alpha u\right|^2 y^k \, dz. \tag{15}$$

We note that a description of the domain of definition of the fractional power $\widetilde{\mathcal{L}}^\alpha$ of the extended operator $\widetilde{\mathcal{L}}$ is given in terms of spaces that were introduced by the author earlier in [5].

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*Note: Figure translations are in progress. See original paper for figures.*