ON CONSTRUCTIVE SETS WITH EQUALITY

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Abstract

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ON CONSTRUCTIVE SETS WITH EQUALITY

AND THEIR MAPPINGS

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The present note belongs to the constructive trend in mathematics (^1).

The term **set** is used here as a synonym for the term "condition with one parameter" ((^2), § 7). If a is a condition with one parameter and x is a constructive object satisfying the condition a, then x is called an **element of the set** a. We shall consider only such sets whose elements are words.

Let a be a set. Suppose that, for the elements of this set, a reflexive, symmetric, and transitive relation has been introduced (by means of a two-place predicate). In this case we shall say that an equality relation has been introduced in the set a, and any two elements of this set that stand in the equality relation will be called **equal**. The pair of objects consisting of the set a and the predicate defining equality of the elements of the set a will be called a **set with equality**, or, briefly, a p**-set**; the set a itself will be called the **base** of this p-set.

Let $\mathcal{A}$ be a p-set. By **elements** of this p-set we shall mean the elements of the base of this p-set. If x and y are elements of the p-set $\mathcal{A}$, then the notation $x = y$ will mean that x and y are equal elements.

The concepts of a set and a p-set are made precise with the aid of some definite logical-mathematical language. The questions considered in this note do not require a compulsory fixation of the language. In those cases where we shall assert the possibility of constructing p-sets satisfying certain conditions, we shall have in mind p-sets for which the base and the condition defining equality of elements are expressible by means of the languages described in §§ 3 and 8 of work (^2).

Let $\mathcal{A}$ and $\mathcal{B}$ be p-sets and let $\mathfrak{A}$ be a normal algorithm (^3). We shall call $\mathfrak{A}$ a **partial mapping** of $\mathcal{A}$ into $\mathcal{B}$ if, for every element x of the p-set $\mathcal{A}$, from $!\mathfrak{A}(x)$ it follows that $\mathfrak{A}(x) \in \mathcal{B}$. Let $\mathcal{A}$ and $\mathcal{B}$ be p-sets and let $\mathfrak{A}$ be a partial mapping of $\mathcal{A}$ into $\mathcal{B}$. We shall call $\mathfrak{A}$ a **total mapping** if the algorithm $\mathfrak{A}$

is applicable to every element of the p-set $\mathcal{A}$. We shall call $\mathfrak{A}$ an **equality-preserving mapping** if, for all elements x and y of the p-set $\mathcal{A}$, from $!\mathfrak{A}(x)$, $!\mathfrak{A}(y)$, and $x = y$, it follows that

$$\mathfrak{A}(x) = \mathfrak{A}(y).$$

We shall call $\mathfrak{A}$ a **univalent mapping** if, for any elements x and y of the p-set $\mathcal{A}$ that are not equal, from $!\mathfrak{A}(x)$ and $!\mathfrak{A}(y)$ it follows that the elements $\mathfrak{A}(x)$ and $\mathfrak{A}(y)$ of the p-set $\mathcal{B}$ are likewise not equal. We shall call $\mathfrak{A}$ a **mapping of the p-set $\mathcal{A}$ onto the p-set $\mathcal{B}$** if, for every element x of the p-set $\mathcal{B}$, there cannot fail to exist an element y of the p-set $\mathcal{A}$ such that $!\mathfrak{A}(y)$ and

$$\mathfrak{A}(y) = x.$$

Mappings preserving equality are considered quite often in constructive mathematics. Examples of such mappings are constructive functions of a real variable (in the sense of A. A. Markov ($\frown$4)); mappings preserving equality and satis—

satisfying certain additional conditions are homomorphisms and isomorphisms of associative calculi [3] (with the equivalence relation of words in the corresponding associative calculi taken as the equality relation of words), etc.

In the present note a classification of p-sets is described according to certain properties of their bases and of the conditions determining equality of elements, and some results are given concerning mappings of p-sets that preserve equality. These results are formulated in terms of the classification of p-sets described here.

Let $\mathcal{A}$ be a p-set. We shall say that $\mathcal{A}$ is **decidable** if the basis of $\mathcal{A}$ is a decidable set. We shall say that $\mathcal{A}$ is **enumerable** if the basis of $\mathcal{A}$ is an enumerable set. We shall call $\mathcal{A}$ **conditionally enumerable** if there exists an enumerable set β such that every element of the set β is an element of the p-set $\mathcal{A}$, and for every element x of the p-set $\mathcal{A}$ there cannot fail to exist an element y of this p-set such that $y \in \beta$ and $x = y$. We shall say that $\mathcal{A}$ has a **decidable equality condition** if there exists a normal algorithm applicable to every pair of elements of the p-set $\mathcal{A}$ and recognizing pairs consisting of equal elements. If x is an element of the p-set $\mathcal{A}$, then by $R(x)$ we shall denote a set such that: a) every element of the set $R(x)$ is an element of the p-set $\mathcal{A}$ equal to the element x; b) every element of the p-set $\mathcal{A}$ equal to the element x is an element of the set $R(x)$. We shall say that the p-set $\mathcal{A}$ has a **completely enumerable equality condition** if for every element x of this p-set the set $R(x)$ is enumerable. We shall say that the p-set $\mathcal{A}$ has a **normal equality condition** if, for all elements x and y of this p-set, the judgment "x and y are equal" is equivalent to its double negation.

We shall say that the p-set $\mathcal{A}$ is **infinite** if there does not exist a finite collection of elements of this p-set that would satisfy the condition: for every element x

of the p-set $\mathcal{A}$ there cannot fail to exist an element equal to x and belonging to this collection. Here we shall confine ourselves to considering only infinite p-sets.

By **types** of p-sets we shall mean words of the form $[a, b]$, where as a one takes the letter P, Π, H, or the word , and as b one takes one of the symbols $\mathrm{P} =$, $\Pi =$, $\Pi \neq$, $\ =$, or $\mathrm{H} =$.

Let $\mathcal{A}$ be a p-set. We shall say that $\mathcal{A}$ **has type** $[a, b]$ if $\mathcal{A}$ is decidable and a is P, or $\mathcal{A}$ is enumerable and a is Π, or $\mathcal{A}$ is conditionally enumerable and a is , or a is H and it is false that $\mathcal{A}$ is conditionally enumerable, and if $\mathcal{A}$ has a decidable equality condition and b is $\mathrm{P} =$, or $\mathcal{A}$ has an enumerable equality condition (5) and b is $\Pi =$, or $\mathcal{A}$ has an enumerable inequality condition (5) and b is $\Pi \neq$, or $\mathcal{A}$ has a completely enumerable equality condition and b is $\ =$, or b is $\mathrm{H} =$ and it is false that $\mathcal{A}$ has at least one of the just-mentioned equality or inequality conditions.

The same p-set may have several different types. If T and U are types of p-sets, then by $T \to U$ we shall denote the judgment: "every infinite p-set having type T also has type U," and by $T \sim U$ we shall denote the judgment: "every infinite p-set having one of the types T, U also has the other of these types." The following relations between types of infinite p-sets hold (here, in place of a and b, one may substitute any symbols allowed for them according to the definition of types of p-sets):

$$[\mathrm{P}, b] \to [\Pi, b], \qquad [\Pi, b] \to [\ , b], \qquad [a, \mathrm{P} =] \to [a, \Pi =],$$

$$[a, \mathrm{P} =] \to [a, \Pi \neq], \qquad [a, \ =] \to [a, \Pi =],$$

$$[\mathrm{P}, \Pi =] \sim [\mathrm{P}, \ =], \qquad [\Pi, \Pi =] \sim [\Pi, \ =] \sim [\ , \ =].$$

Considering concrete examples of p-sets, one can show that no other relations of the indicated kind between the types of infinite p-sets exist.

Theorem 1. Let $\mathcal{A}$ and $\mathcal{B}$ be p-sets, and let $\mathcal{A}$ be conditionally enumerable. If there exists a total mapping of $\mathcal{A}$ onto $\mathcal{B}$ that preserves equality, then $\mathcal{B}$ is also conditionally enumerable.

Theorem 2. Let $\mathcal{A}$ and $\mathcal{B}$ be infinite p-sets, and let $\mathcal{A}$ have type $[\ , \mathrm{P} =]$, while $\mathcal{B}$ is conditionally enumerable. Then one can construct a total mapping of $\mathcal{A}$ onto $\mathcal{B}$ that preserves equality. If the p-set $\mathcal{B}$ also has type $[\ , \mathrm{P} =]$, then such a mapping can be constructed one-to-one.

Let $\mathcal{A}$ and $\mathcal{B}$ be p-sets and let $\mathfrak{A}$ be a partial mapping of $\mathcal{A}$ into $\mathcal{B}$ preserving equality. We shall call $\mathfrak{A}$ a **pretotal mapping** if, for every element x of the p-set $\mathcal{A}$, there cannot fail to exist an element of this p-set equal to it to which the algorithm $\mathfrak{A}$ is applicable. We shall say that $\mathfrak{A}$ is **extendable to a total mapping preserving equality** if one can construct a total mapping $\mathfrak{B}$ of the p-set $\mathcal{A}$ into the p-set $\mathcal{B}$, preserving equality, such that for every element x of the p-set $\mathcal{A}$, if $!\mathfrak{A}(x)$, then $\mathfrak{A}(x) = \mathfrak{B}(x)$.

Theorem 3. Let $\mathcal{A}$ be a p-set having a completely enumerable equality condition. Then every pretotal mapping of the p-set $\mathcal{A}$ into any p-set $\mathcal{B}$, preserving equality, is extendable to a total mapping preserving equality.

Let $\mathcal{A}$ and $\mathcal{B}$ be p-sets and let $\mathfrak{A}$ be a partial mapping of $\mathcal{A}$ into $\mathcal{B}$. By a mapping **inverse to** $\mathfrak{A}$ we shall mean any partial mapping $\mathfrak{B}$ of the p-set $\mathcal{B}$ into the p-set $\mathcal{A}$ such that, for every element x of the p-set $\mathcal{B}$: a) if $!\mathfrak{B}(x)$, then $!\mathfrak{A}(\mathfrak{B}(x))$ and $\mathfrak{A}(\mathfrak{B}(x)) =_B x$; b) if there exists an element y of the p-set $\mathcal{A}$ such that $!\mathfrak{A}(y)$ and $\mathfrak{A}(y) =_B x$, then $!\mathfrak{B}(x)$. Let us note that an algorithm inverse to the algorithm $\mathfrak{A}$ in the sense defined in (6) is not always a mapping inverse to $\mathfrak{A}$.

Theorem 4. Let $\mathcal{A}$ be a conditionally enumerable p-set having a normal equality condition, and let $\mathcal{B}$ be a p-set having an enumerable equality condition. Then, for every total one-to-one mapping $\mathfrak{A}$ of the p-set $\mathcal{A}$ into the p-set $\mathcal{B}$, preserving equality, one can construct a mapping inverse to $\mathfrak{A}$ and preserving equality.

Theorem 5. Let $\mathcal{A}$ be a conditionally enumerable p-set and let $\mathcal{B}$ be a p-set having a decidable equality condition. Then, for every total mapping $\mathfrak{A}$ of the p-set $\mathcal{A}$ into the p-set $\mathcal{B}$, preserving equality, one can construct a mapping inverse to $\mathfrak{A}$ and preserving equality.

Let $\mathcal{A}$ and $\mathcal{B}$ be p-sets. We shall say that $\mathcal{A}$ is **superposable on** $\mathcal{B}$ if there exists a total one-to-one mapping of $\mathcal{A}$ onto $\mathcal{B}$ preserving equality. We shall say that $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** if there exists a total mapping $\mathfrak{A}$ of the p-set $\mathcal{A}$ onto the p-set $\mathcal{B}$, and there exists a total mapping $\mathfrak{B}$ of the p-set $\mathcal{B}$ onto the p-set $\mathcal{A}$, both one-to-one and preserving equality, such that $\mathfrak{A}$ is a mapping inverse to $\mathfrak{B}$, and $\mathfrak{B}$ is a mapping inverse to $\mathfrak{A}$.

One can construct an example of p-sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is superposable on $\mathcal{B}$, $\mathcal{B}$ is superposable on $\mathcal{A}$, but $\mathcal{A}$ and $\mathcal{B}$ are not equivalent.

By the **principal types** of p-sets we shall mean types $[a, b]$, where a is taken to be or , and b is taken to be P $=$, $=$, $\neq$, or $=$. One and the same p-set may have several different principal types.

Theorem 6. Let $\mathcal{A}$ and $\mathcal{B}$ be p-sets and let T be a principal type. If $\mathcal{A}$ and $\mathcal{B}$ are equivalent and $\mathcal{A}$ has type T, then $\mathcal{B}$ has type T.

Theorem 7. Any two infinite p-sets having type $[\mathrm{UP}, P =]$ are equivalent.

For every principal type T of p-sets distinct from the type $[\mathrm{UP}, P =]$, one can construct p-sets having type T and not equivalent.

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Note: Figure translations are in progress. See original paper for figures.