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ON THE NUMBER OF EDGES IN A GRAPH WITH GIVEN RADIUS

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In this note the maximum number of edges in an n-vertex graph having a given radius is established. Only finite connected undirected graphs without loops and parallel edges are considered ([1]).

The **distance** $\rho(x, y)$ between vertices x and y is the length (the number of edges) of the shortest chain joining x and y. The **radius** $r(G)$ of a graph G is the greatest integer possessing the following property: whatever vertex x of the graph G is taken, there is a vertex y such that $\rho(x, y) \geq r(G)$. The **diameter** $d(G)$ of a graph G is the greatest distance between two vertices of the graph G.

Obviously, $r(G) \leq d(G)$. If G is a tree, then

$$r(G) = [(d(G) + 1)/2].$$

It follows from this, in particular, that the number of vertices in a graph is at least twice its radius.

We shall denote by $\sigma(x)$ the degree of the vertex x, and by $\sigma(G)$ the maximum degree of a vertex of the graph G.

Lemma 1. In an n-vertex graph G,

$$\sigma(G) \leq n - 2r(G) + 2.$$

Lemma 2. Let G be an n-vertex graph with $r(G) \geq 3$, and let x and y be vertices of G with $\rho(x, y) \geq 3$. Then

$$\sigma(x) + \sigma(y) \leq n - 2r(G) + 4.$$

The proofs of Lemmas 1 and 2 are analogous.

We prove Lemma 2. Let the vertices $x_1, x_2, \dots, x_i$ be adjacent to the vertex x, and the vertices $y_1, y_2, \dots, y_j$ to the vertex y. Construct a tree T covering the graph G and containing the edges

$$(x, x_1), \dots, (x, x_i),\ (y, y_1), \dots, (y, y_j).$$

It must be that

$$d(T) \geq 2r(G) - 1.$$

Take an elementary chain joining two most distant vertices of the tree T. This chain contains no more than 6 vertices from the set

$$x, x_1, \dots, x_i;\ y, y_1, \dots, y_j$$

and no fewer than $2r(G) - 6$ other vertices. Consequently,

$$n \geq 2r(G) - 6 + i + 1 + j + 1,$$

whence

$$\sigma(x) + \sigma(y) = i + j \leq n - 2r(G) + 4,$$

as was required to prove.

Let n and k be natural numbers and let $n \geq 2k \geq 2$. Denote by $f(n,k)$ the maximum number of edges that an n-vertex graph with radius k may have, and by $C(n,k)$ an n-vertex graph with $f(n,k)$ edges and with $r(C(n,k)) = k$. Obviously,

$$f(n+1,k) > f(n,k)$$

and

$$f(n,k+1) < f(n,k).$$

Theorem.

$$f(n,1) = \frac{n(n-1)}{2}, \qquad f(n,2) = \left[\frac{n(n-2)}{2}\right];$$

for $k \geq 3$,

$$f(n,k) = \frac{n^2 - 4kn + 5n + 4k^2 - 6k}{2}.$$

Proof. The first two equalities are obvious. Now let $k \geq 3$. For brevity denote

$$g(n,k) = \frac{n^2 - 4kn + 5n + 4k^2 - 6k}{2} \qquad (n \geq 2k \geq 6).$$

We first show that $f(n,k) \geq g(n,k)$. Indeed, let the graph H have the following structure: its vertices are

$$a_1, a_2, \dots, a_{2k},\ b_1, \dots, b_{n-2k};$$

the subgraph generated by the vertices $a_1, a_2, \dots, a_{2k}$ is the elementary cycle

$$[a_1, a_2, \dots, a_{2k}, a_1];$$

the vertices $b_1, \dots, b_{n-2k}$ are pairwise adjacent, and each of them is adjacent to the vertices a_1, a_2, a_3. The n-vertex graph H has, obviously, $g(n,k)$ edges and $r(H) = k$. This proves the required inequality.

We shall prove the inequality $f(n,k) \leqslant g(n,k)$ by double induction on n and k.

Basis of the induction:

1) $f(n,3) \leqslant g(n,3)$ for every $n \geqslant 6$.

2) $f(2k,k) \leqslant g(2k,k)$ for every $k \geqslant 3$.

3) It is not difficult to verify that in the graph $C(n,3)$ one can always indicate three nonintersecting pairs of vertices $\{x_1, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$ such that $\rho(x_i, y_i) \geqslant 3$ $(i = 1, 2, 3)$. By Lemmas 1 and 2, then

$$f(n,3) \leqslant \frac{[3(n-2) + (n-6)(n-4)]}{2} = \frac{(n^2 - 7n + 18)}{2} = g(n,3).$$

2) This is obvious.

Induction step. Let $k \geqslant 4$, $n \geqslant 2k+1$, and suppose it has been proved that $f(n_0, k_0) \leqslant g(n_0, k_0)$ if either $3 \leqslant k_0 < k$, or $k_0 = k$ but $n_0 < n$. Consider the subgraph L of the graph $C(n,k)$ generated by all vertices except one vertex x that is not a cut point. If $r(L) \geqslant k$, then, by the induction hypothesis, the graph L contains no more than $g(n-1,k)$ edges, and since, by Lemma 1, $\sigma(x) \leqslant n - 2k + 2$, we have $f(n,k) \leqslant g(n-1,k) + n - 2k + 2 = g(n,k)$.

Suppose now that any subgraph of the graph $C(n,k)$ obtained by deleting from $C(n,k)$ a vertex different from a cut point has radius $k-1$. Then for every vertex x of the graph $C(n,k)$ that is not a cut point there is a vertex y such that $\rho(y,x) = k$, but for $z \neq x$, $\rho(y,z) \leqslant k-1$. We shall call the vertex y conjugate to the vertex x. Obviously, if y is also not a cut point, then the vertex x is conjugate to the vertex y.

Taking into account the indicated assumption on the structure of the graph $C(n,k)$, as well as the induction hypothesis, it is not difficult to prove the validity of the inequality $f(n,k) \leqslant g(n,k)$ in the case where the graph has at least one cut point. Therefore we shall assume that the graph $C(n,k)$ has no cut points.

Since $d(C(n,k)) \geqslant k$, by Menger' s theorem ([1]), in the graph $C(n,k)$ there exists an elementary cycle of length $\geqslant 2k$. Among all elementary cycles of length $\geqslant 2k$, choose a cycle M_0 of least length $m_0 \geqslant 2k$. The cycle M_0 has no chords, i.e., the subgraph generated by the vertices of the cycle M_0 is an elementary cycle. Indeed, otherwise in the graph $C(n,k)$ one could indicate a set consisting of $2k-2$ vertices, the distance between any two of which does not exceed $k-1$. Since to each of these vertices there is mutually uniquely conjugate some vertex of the graph $C(n,k)$, by Lemmas 1 and 2,

$$f(n,k) \leqslant \frac{[(2k-2)(n-2k+4) + (n-2k+2)(n-4k+4)]}{2} < g(n,k),$$

which is impossible.

Let x be a vertex not belonging to the cycle M_0. If a vertex of the cycle M_0 is conjugate to the vertex x, then, obviously, x is adjacent to no more than $m_0 - 2k + 3$ vertices of the cycle M_0. If, however, a vertex y not lying on the cycle

M_0 is conjugate to the vertex x, then, having no common adjacent vertices, the vertices x and y are adjacent in total to no more than $m_0-2k+6 \leqslant 2(m_0-2k+3)$ vertices of the cycle M_0. Thus,

$$f(n,k) \leqslant m_0 + (n-m_0)(m_0-2k+3) + \frac{(n-m_0)(n-m_0-1)}{2} \leqslant g(n,k).$$

The theorem is proved.

Let us note in conclusion that for fixed n and k satisfying the conditions $k \geqslant 3$, $n \geqslant 2k+2$, the graphs $C(n,k)$ need not be isomorphic.

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REFERENCES

1. C. Berge, *Graph Theory and Its Applications*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.