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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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I. N. GORSHKOVA, K. P. STANYUKOVICH

THE INITIAL STAGE OF A TWO-DIMENSIONAL UNSTEADY GAS FLOW

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Let us consider the problem of a plane unsteady outflow of a compressed gas from a tube. We shall assume that the gas obeys the polytropic equation

$p = \text{const} \cdot \rho^k.$

Suppose that at the instant $t = 0$ a wall is removed and the gas begins to flow out. It is obvious that the gas flow will be isentropic and nonstationary. In this case a rarefaction wave will propagate through the gas from right to left, its front being parallel to the y-axis (see Fig. 1). The velocity of the front is equal to $-c_n$, where c_n is the sound speed in the undisturbed gas. At the time $t = l/c_n$, where l is the length of the tube, this wave will reach the left wall of the tube. In the case of gas outflow into a plane tube, its motion is described by the equations of one-dimensional unsteady gas motion. In our case, however, expansion of the gas to the sides will take place. The expansion will initially occur in two lateral rarefaction waves.

Fig. 1

Let us determine the change in the position of the fronts of these waves with time. It is obvious that these fronts of the lateral rarefaction waves will be characteristics of the gas-dynamic equations

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{2}{k-1}c\frac{\partial c}{\partial x} = 0,\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{2}{k-1}c\frac{\partial c}{\partial y} = 0,\\
&\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + \frac{k-1}{2}c\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.
\end{aligned}
\tag{1}
$$

The equation of the characteristics $f(t; x; y) = 0$ then, as is known, has the form

([1])

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = c\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}. \tag{2}$$

Since in our case the front of the lateral rarefaction wave will adjoin the region of one-dimensional motion, where the velocity component along the y-axis is $v = 0$, the characteristic equations take the form

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = c\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}. \tag{3}$$

Owing to the continuity of the flow parameters as a rarefaction wave passes, the values of u and c on the fronts of the lateral rarefaction waves must

be equal to the values of u and c in the one-dimensional unsteady "Riemann" flow, for which

$$u - c = \frac{x}{t}, \qquad u = \frac{2}{k-1}\left(c_{\mathrm{H}} - c\right). \tag{4}$$

Since the motion in the initial rarefaction wave is self-similar and depends on one independent variable $z_1 = x/t$, in our case, because the lateral dimensions of the tube do not enter into the determination of the flow parameters in the lateral rarefaction wave, the parameters of this wave will depend on two independent variables $z_1 = x/t$, $z_2 = y/t$.

Equations (1) in these variables take the form

$$\begin{gathered}\frac{\partial u}{\partial z_1}(u - z_1) + \frac{\partial u}{\partial z_2}(v - z_2) + \frac{2c}{k-1}\frac{\partial c}{\partial z_1} = 0,\\ \frac{\partial v}{\partial z_1}(u - z_1) + \frac{\partial v}{\partial z_2}(v - z_2) + \frac{2c}{k-1}\frac{\partial c}{\partial z_2} = 0,\\ \frac{\partial c}{\partial z_1}(u - z_1) + \frac{\partial c}{\partial z_2}(v - z_2) + \frac{k-1}{2}c\left(\frac{\partial u}{\partial z_1} + \frac{\partial v}{\partial z_2}\right) = 0.\end{gathered} \tag{5}$$

Gas motions described by equations (5) will be self-similar. Equation (3) will now take the form

$$-\frac{\partial f}{\partial z_1}(u - z_1) + \frac{\partial f}{\partial z_2}z_2 = c\sqrt{\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2},$$

and after transformations, since

$$df = \frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial z_2} dz_2, \qquad u - z_1 = c,$$

it can be rewritten in the form

$$\frac{dz_2^2}{dz_1} = -\frac{z_2^2 - c^2}{c} = -\frac{z_2^2}{c} + c = -\frac{(k+1)z_2^2}{2\left[c_\mathrm{H} - \frac{k-1}{2}z_1\right]} + \frac{2}{k+1}\left(c_\mathrm{H} - \frac{k-1}{2}z_1\right). \quad (6)$$

The solution of the resulting equation under the condition $z_{01} = 0,\ z_{02} = 0$ (see Fig. 1) takes the form

$$z_2^2 = \left(z_1 - \frac{2}{k-1}c_\mathrm{H}\right)^{(k+1)/(k-1)} \frac{(k-1)^2}{(3-k)(k+1)} \left[\left(z_1 - \frac{2c_\mathrm{H}}{k-1}\right)^{-(3-k)/(k-1)} - \left(-\frac{2c_\mathrm{H}}{k-1}\right)^{-(3-k)/(k-1)}\right]. \quad (7)$$

In the case $k = 3$, solution (7) takes the form

$$z_2^2 = \frac{1}{2}(z_1 - c_\mathrm{H})^2 \ln \frac{c_\mathrm{H}}{c_\mathrm{H} - z_1}, \quad (8)$$

$$z_1 = z_1^* = \frac{2}{k-1}c_\mathrm{H}\left\{1 - \left[\frac{k+1}{2(k-1)}\right]^{-(k-1)/(3-k)}\right\}, \quad (9)$$

or, for $k = 3$, with $\ln(c_\mathrm{H} z_1^*/c_\mathrm{H}) = -1/2$,

$$z_1^* = 0.39c_\mathrm{H}, \qquad z_2^* = 0.30c_\mathrm{H}. \quad (10)$$

The value $z_2 = z_2^*$ becomes minimal for the upper lateral rarefaction wave.

On the basis of an analysis of the data obtained, the pattern of the unsteady gas flow in the case under consideration is shown in Fig. 1.

The unsteady gas flow at the initial instant may be represented in the following way: in the central zone there is the so-called Riemann wave, the principal relations between the parameters of which are known from the theory of one-dimensional unsteady flow; below and above are located the lateral rarefaction waves.

The obtained solutions are valid up to the time t_1, at which the two lateral rarefaction waves meet at the point where $z_2 = z_2^* = z_{2\,\mathrm{min}}$ (for the upper wave), with $z_2^* = -a/2t_1$, where a is the lateral dimension of the tube. The instant of merging of these two fronts of the lateral rarefaction waves corresponds to the time at which the conditions are simultaneously satisfied:

$$z = -\frac{a}{2t_1} = z_2^*, \qquad z_1 = z_1^* = \frac{2c_{\text{H}}}{k-1}\left\{1 - \left[\frac{k+1}{2(k-1)}\right]^{-(k-1)/(3-k)}\right\};$$

which gives

$$\frac{a}{2t_1} = \frac{2c_{\text{H}}}{k-1}\left(\frac{k-1}{k+1}\right)^{2/(3-k)} 2^{(k-1)/(3-k)}, \tag{11}$$

which determines the value t_1 and, further, z_2^* and $x^* = z_1^* t_1$.

Similarly, for $k = 3$,

$$\frac{a}{2t_1} = (c_{\text{H}} - z_1^*)\left[\frac{1}{2}\ln\frac{c_{\text{H}}}{c_{\text{H}} - z_1}\right]^{1/2} = 0.30c_{\text{H}}. \tag{12}$$

After these rarefaction waves meet, a new "median" rarefaction wave arises between them, symmetric about the x-axis at $y = -a/2$, which can be described by the general solution. This solution will not be self-similar. Approximately, it may be sought by assuming that the parameters of this wave do not depend on y, but only on x and t.

Let us now find the "external" characteristics, or the outer boundaries of the expanding gas. For this purpose we transform the system of equations (5), introducing a velocity potential, since isentropic gas motion is always potential. In this case

$$u = \partial\varphi/\partial z_1, \qquad v = \partial\varphi/\partial z_2, \qquad \varphi = \varphi^*/t,$$

where φ^* is the ordinary potential;

$$uz_1 + vz_2 = \varphi + \frac{c^2}{k-1} + \frac{u^2+v^2}{2}, \tag{13}$$

which is Bernoulli' s equation for the class of flows under consideration. Here $c^2/(k-1) = i$ is the heat content.

Interchanging the dependent and independent variables, after the usual transformations, using the continuity equation, we arrive at the equation

$$\frac{\partial z_2}{\partial u}\left[(u-z_1)^2 - c^2\right] + \frac{\partial z_1}{\partial v}\left[(v-z_2)^2 - c^2\right] = 2\frac{\partial z_2}{\partial u}(u-z_1)(v-z_2), \tag{14}$$

where $z_1 = \partial F/\partial u,\ z_2 = \partial F/\partial v$, and F is the new potential function, so that

$$i + \frac{u^2+v^2}{2} = F. \tag{15}$$

Finally, one can obtain a single second-order equation, which is a generalization of Chaplygin' s equation for the given class of gas flows:

$$\frac{\partial^2 F}{\partial u^2}\left[\left(v-\frac{\partial F}{\partial v}\right)^2-c^2\right]-2\frac{\partial^2 F}{\partial u\,\partial v}\left(u-\frac{\partial F}{\partial u}\right)\left(v-\frac{\partial F}{\partial v}\right)+\frac{\partial^2 F}{\partial v^2}\left[\left(u-\frac{\partial F}{\partial u}\right)^2-c^2\right]=0, \tag{16}$$

where

$$c^2=(k-1)\left(F-\frac{u^2+v^2}{2}\right).$$

Near the origin of coordinates (near the point $z_1=0;\ z_2=0$), since $\partial F/\partial u\to 0,\ \partial F/\partial v\to 0$, this equation becomes the ordinary Chaplygin equation (2), with $c^2=(k-1)\left(F_0-(u^2+v^2)/2\right)$, where $F_0=\text{const}$. Since for $x=0$, $u_k=c_k=2c_{\text{H}}/(k+1)$, taking a reference frame in which the gas in the section $x=0$ is at rest, we may use the Prandtl-Meyer solution to investigate the lateral outflow (3) near the point $x=0,\ y=0$. All equations describing the motion of the gas, and

the continuity equations remain the same if one sets $x\to x-u_k t$, $u\to u-u_k$, $z\to z-u_k$. In this reference frame the external line on which $c=0$ will be a circle of radius $z^*=4c_{\text{H}}/(k^2-1)$; its equation will be

$$\left(z_1-\frac{2c_{\text{H}}}{k+1}\right)^2+z_2^2=\left(\frac{4c_{\text{H}}}{k^2-1}\right)^2. \tag{17}$$

The Prandtl-Meyer solution for the leftmost limiting characteristic, on which $c=0$, gives the values of the "fan" angle

$$\sigma=\operatorname{arc\,tg}\frac{v^*}{u^*}=\frac{\pi}{2}\left(\sqrt{\frac{k+1}{k-1}}-1\right), \tag{18}$$

where

$$\sqrt{u^{*2}+v^{*2}}=4c_{\text{H}}/(k^2-1).$$

From this it is easy to find u^* and v^*. Since the equations of the characteristics for $c=0$ are

$$\frac{dz_2}{dz_1}=\frac{z_2-v}{z_1-u}\quad\text{or}\quad z_2-v=\frac{v^*}{u^*}(z_1-u),$$

then for $u=u^*$, $v=v^*$,

$$z_2=\frac{v^*}{u^*}z_1,$$

which determines the equation of the rectilinear boundary of the gas fan. The complete solution for the "lateral" wave must be sought with the aid of equation (16), which, however, is a difficult problem and can be solved approximately.

In conclusion it is interesting to note that only as $t \to \infty$ will the right-hand part of the gas obey the Riemann solution and be "swept away" by the subsequent waves; at finite times elapsed after the onset of the fan, this part will exist.

Research Institute of Introscopy

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.