On the problem of the analytic construction of regulators

Authors: V. I. Bondarenko, Yu. M. Filimonov

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Abstract

The paper considers the well-known problem of the analytical design of a controller for a linear controlled system. However, in contrast to existing developments, this study examines a more complex nonlinear case rather than a quadratic optimality criterion. For the aforementioned criterion, the problem of constructing an optimal control action satisfying the constraint $|u^0(\tau)| \leq 1$ is investigated. The problem is solved using a limit transition. Bibliography: 5 items.

Full Text

Introduction

This work considers a control system described by an n-dimensional vector differential equation:

$$\dot{x} = Ax + bu, \quad (b > 0) \tag{1.1}$$

where $u(t)$ is a scalar control function subject to the constraint $|u(t)| \leq 1$ (1.5). The system matrix A and the vector b are defined such that the components of the system satisfy specific structural properties. Specifically, we assume the coefficients a_k for $k = 1, \dots, n$ and the off-diagonal elements a_{is} (where $s = i-1$) satisfy the condition $a_{is} > 0$ (1.4).

The objective is to minimize a quadratic-type performance functional:

$$J(x_0, u) = \int_0^\infty [u^2 + \alpha F^2(x)]\, dt \tag{1.6}$$

where α is a positive weighting parameter and $F(x)$ is a scalar function of the state vector. We assume $F(x)$ is a continuously differentiable function satisfying the monotonicity conditions:

$$\frac{\partial F}{\partial x_1} > 0, \quad \frac{\partial F}{\partial x_m} > 0 \quad (m = 2, \dots, n) \tag{1.7}$$

Furthermore, we require that $F(0) = 0$.

Optimal Control Synthesis

Let $u_0(x)$ be the optimal control law for the system (1.1) under the constraint (1.5). The state space is partitioned into regions Q^+ and Q^- by a switching surface S. The control law is defined as:

$$u_0(x) = \begin{cases} +1, & x \in Q^+ \\ -1, & x \in Q^- \\ 0, & x \in S \end{cases} \tag{2.1}$$

For a given initial state x_0, the optimal trajectory $x^0(t)$ and the corresponding optimal control $u^0(t)$ must satisfy the optimality condition:

$$\int_0^{T_\epsilon} [u_a^2 + \alpha F^2(x_a^0)]\, dt < \int_0^{T_\epsilon} [u_0^2 + \alpha F^2(x^0)]\, dt \tag{2.3}$$

where u_a represents an admissible control and x_a^0 is the resulting trajectory. A key requirement for the stability and optimality of the system is that the difference in the integrands remains positive, specifically:

$$F^2(x_a^0) - F^2(x^0) > 0 \tag{2.4}$$

Stability and Convergence Analysis

To analyze the behavior of the system near the switching surface S, we consider an ϵ-neighborhood defined by $\|x\|^2 < \epsilon$. Within this neighborhood, the control $u_0(x)$ effectively governs the system' s approach to the equilibrium. Using the properties of the matrix A and the positivity conditions in (1.4), we can express the state components through integral representations. For instance, the second state component can be represented as:

$$x_2 = a_{21} \int_0^t e^{a_{11}(t-\tau)} u(\tau)\, d\tau \tag{2.6}$$

Given the constraints on the coefficients a_{ik}, it follows that for $x_0 \in Q^-$, the trajectory $x^0(t)$ with $u^0(t) = -1$ leads to $x_2 > 0$ under appropriate conditions (2.7).

By applying the mean value theorem to the performance functional, the difference in the potential function can be approximated as:

$$F^2(x_a^0) - F^2(x^0) \approx 2F(x^0)\Delta F \tag{2.8}$$

As $\alpha \to \infty$, the control law $u_0(x)$ converges to a bang-bang structure. The sequence of trajectories $\{x_m^0(t)\}$ and controls $\{u_m^0(t)\}$ converges to the optimal pair (x^*, u^*) in the limit. This convergence is supported by the fact that the integral of the control effort remains bounded:

$$\int_0^{T^0} u_a^2(t)\,dt < T^0 \tag{2.9}$$

Finally, we observe that the difference in the functional values:

$$\int [F^2(x^*) - F^2(x^0)]\,dt \leq \ldots \tag{2.11}$$

tends to zero as the trajectory approaches the optimal path. This confirms that the synthesized control law (2.1) provides an asymptotically stable solution that minimizes the cost functional (1.6) while satisfying the physical constraints of the system.

References

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Note: Figure translations are in progress. See original paper for figures.