Limit of the solution to the wave equation for an inhomogeneous medium as $t \to \infty$

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Abstract

The non-stationary problem

$$u_{xx} = k^2(x)u_{tt}, \tag{1}$$

is investigated, where $k(x) = k_0$ for $x < 0$ and $k(x) = k_1$ for $x > x_0$, under the initial condition $u_0(x,t) = \mu(t - k_0 x)$ for $t < 0$, where $\mu(z) = 0$ for $z < 0$.

It is shown that under the condition $\operatorname{var} \ln k(x) < \pi$, the limit of the solution to equation (1) as $t \to \infty$ for the case of an incident wave of the form

$$u_0(x,t) = \mu(t - k_0 x), \quad \mu(z) = \mu_0, \quad z > z_0$$

is equal to $\lim_{t\to\infty} u(x,t) = \frac{2\mu_0 k_0}{k_0 + k_1}$. That is, the limit is the same as in the case where $k(x) = k_0$ for $x < 0$ and $k(x) = k_1$ for $x > 0$.

This result is generalized to the case where $\lim_{x\to-\infty} k(x) = k_0$ and $\lim_{x\to+\infty} k(x) = k_1$. Previously, Atkinson obtained a similar result for the stationary problem $u_{xx} - k^2(x)u = 0$ under the condition $\operatorname{var} \ln k(x) \le \pi$.

Full Text

Preamble

In 1967, I. Z. Kayaks [1] investigated the equation $U_{xx} + k^2(x)U = 0$ for $0 < t < a$, as discussed in [2]. Considering the limit as $t \to 0$, where $\mu(z) = 0$ and $n - k_0 x$, it was shown that if $\operatorname{var} \ln k(x) < \infty$ as $n \to -\infty$, the solution behaves according to the conditions established in [1]. As $x \to +\infty$, the function $k(x)$ is such that the initial wave $U_0(x,t) = \mu(t - k_0 x)$ for $t < 0$, with $\mu(z) = \mu_0 = \text{const}$ for $z > 0$. For the wave equation $U_{xx} = k^2(x)U_{tt}$, we assume $k(x) = k_1$ for $x < 0$. As shown in [FIGURE: 1], we assume $k(x)$ varies such that for $x > x_0$, $k(x) = k_0 = \text{const}$, and the derivative $k'(x)$ is proportional to $1/k_0$. For $t < 0$, the solution is $U(x,t) = \mu(t - k_0 x)$, where $\mu(z) = 0$ for $z < 0$.

Following the methodology in [2], the solution to the system (1) can be represented as the sum of two components, $v^*(x,t)$ and $W^*(x,t)$, which satisfy the following integral relations:

$$v^*(x,t) = v_0(x,t) + \frac{1}{2k(x)^{1/2}} \int k'(s) W^* \left[s, \int k(l)dl \right] ds,$$
$$W^*(x,t) = \frac{1}{2k(x)^{1/2}} \int k'(s) k(s) \left[v^* \left(s, \int k(l)dl \right) \right] ds.$$

The derivatives are related by $k(x)v_t^*(x,t) = -v_x^*(x,t) + \frac{1}{2}[W^*(x,t) - v^*(x,t)]$ and $k(x)W_t(x,t) = W_x^*(x,t) + \frac{1}{2}[W^*(x,t) - v^*(x,t)]$. Defining $v(x,t) = k^{1/2}v^*(x,t)$ and $W(x,t) = k^{1/2}(x)W^*(x,t)$, and introducing the variables $\sigma = \int k(l)dl$ and σ_0, we obtain the system:

$$v(\sigma,t) = k^{1/2}\mu(t-\sigma) + \frac{1}{2} \int \frac{g'(s)}{g(s)} W(s, \sigma + s) ds,$$
$$W(\sigma,t) = -\frac{1}{2} \int \frac{g'(s)}{g(s)} v(s, t + \sigma - s) ds,$$

where $g(\sigma) = k[x(\sigma)]$. These equations can be solved via successive approximations: $v = v_0 + v_2 + v_4 + ...$ and $W = W_1 + W_3 + W_5 + ...$, where the base term is $v_0(\sigma,t) = k^{1/2}\mu(t-\sigma)$.

Under the condition that $|\mu(\sigma)| < M$ and defining $q(\sigma) = \int \frac{|g'(s)|}{g(s)} ds$, we assume $q_0 = q(\sigma_0) < \pi/2$. As demonstrated in [2], if $\operatorname{var} \ln k(x) < \pi$, the terms of the series are bounded by:

$$|v_{2k}(\sigma,t)| \le M k_0^{1/2} \frac{q^{2k}}{(2k)!},$$
$$|W_{2k+1}(\sigma,t)| \le M k_0^{1/2} \frac{q^{2k+1}}{(2k+1)!}.$$

Summing these estimates, we find $|U(t,\sigma)| = |v + W| < M k_0^{1/2} \exp(q)(1 + \tan q_0)$. Specifically, the solution satisfies:

$$|v(\sigma,t)| \le M k_0^{1/2} [\cos q + \alpha \sin q] = M k_0^{1/2} \cos(q_0 - q).$$

In the limit as $t \to \infty$, we define the stationary transmission and reflection coefficients. For the case where $\mu(z) = \mu_0$ for $z > z_0$, the asymptotic values $v_{2k}(\sigma)$ and $W_{2k+1}(\sigma)$ are reached for $t > 2k\sigma_0 - \sigma + z_0$.

The final solution for the transmitted wave $U(\sigma)$ and reflected wave $W(\sigma)$ can be expressed as:

$$U(\sigma) = \mu_0 k_0^{1/2} [\cosh \tau + (\rho - 1) \sinh \tau],$$
$$W(\sigma) = \mu_0 k_0^{1/2} [\sinh \tau + (\rho - 1) \cosh \tau],$$

where $\rho = 1 - \tanh \tau_0$. For a medium where $k(x) = k_0$ for $x < 0$ and $k(x) = k_1$ for $x > x_0$, the reflection coefficient R is given by the standard formula $R = \frac{k_0 - k_1}{k_0 + k_1}$.

In the general case where $\lim_{x \to -\infty} k(x) = k_0$ and $\operatorname{var} \ln k(x) < \pi$ over the interval $(-\infty, +\infty)$, we consider a truncated medium $k_N(x)$ such that $k_N(x) = k(x)$ for $|x| < N$. By taking the limit $N \to \infty$, we ensure the convergence of the approximate solutions v_N and W_N to the exact solutions v and W. The error $|v + W - U_N|$ is bounded by ϵ for sufficiently large N, confirming that the integral representation remains valid for infinite domains provided the total variation of the logarithm of the refractive index is bounded.

References

1. Atkinson, F. V. *Journal of Mathematical Analysis and Applications*, 1, No. 3, 4, 255-276, 1960.
2. [Author Initials]. *Journal of Numerical Physics* (N. F.), No. 4, 66-74, 1964. Submitted May 12, 1966, Moscow State University.

Note: Figure translations are in progress. See original paper for figures.