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Abstract

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ON THE ELEMENTARY THEORY OF LATTICE-ORDERED ABELIAN GROUPS AND K-LINEALS

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According to ([1]), the class of all linearly ordered abelian groups (l.o.a. groups) is decidable, i.e., the elementary theory of this class is decidable. On the other hand, it is not difficult to show that the class of all partially ordered abelian groups is undecidable. In connection with this (and, apparently, independently of it as well), the question of decidability of the class of all lattice-ordered abelian groups (l.o.a. groups) is of interest. In particular, this question was formulated by A. I. Mal' tsev at the Congress of Mathematicians in Moscow in 1966 (see also ([2])). In ([3]) the decidability of the universal theory of the class of all l.o.a. groups is proved, and a classification of l.o.a. groups by universal properties is given.

Let L denote the class of all l.o.a. groups G satisfying the following requirements:

1. It is possible to define multiplication of the elements of G by real numbers, turning G into a K-lineal, i.e., into such a vector space in which multiplication is compatible with the lattice operations; see ([4]). In particular, G is a divisible group.

2. G is Archimedean (see ([5])).

3. The lattice $\hat{G}$ of threads (see ([5])) of the l.o.a. group G is atomic.

4. $\hat{G}$ is a Boolean algebra.

Theorem 1. *Every class of l.o.a. groups containing L is undecidable.*

In the definition of L, atomicity may be replaced by atomlessness. Requirement 4 may be replaced by the requirement that $\hat{G}$ be a relatively complemented lattice without a unit. Finally, requirement 1 may be replaced by the requirements of divisibility and countability. Requirement 1 may also be replaced by the requirement that G be countable and free. In all these variations Theorem 1 remains true.

Which natural classes of l.o.a. groups, then, are decidable? The decidability of some classes of l.o.a. groups follows, according to general theorems of model theory (see ([6])), from the decidability of the class of all l.o.a. groups. There is also the following result.

Theorem 2. *The class of all l.o.a. groups with a finite number of threads is decidable.*

Proof. Let us call a *tree* a finite partially ordered set satisfying the axiom $(x \leq y \,\&\, x \leq z) \to (y \leq z \vee z \leq y)$. From the classification of l.o.a. groups with a finite number of threads by elementary properties, indicated in ([2]), it follows that, up to elementary equivalence, an l.o.a. group with a finite number of threads is nothing other than the generalized product, in the sense of Feferman and Vaught (see ([7]) or ([6])), of certain l.o.a. groups relative to the algebra of subsets of a certain tree. From the Feferman–Vaught theorem it thus follows that the decidability of the class of all l.o.a. groups with a finite number of threads follows from the decidability of the class of all l.o.a. groups and from the decidability of the theory of trees with quantifiers over unary predicates. The latter follows in an obvious way from ([8]).

We note that many natural subclasses of the class of all structurally ordered Abelian groups with a finite number of threads are also decidable: the class of all divisible structurally ordered Abelian groups with a finite number of threads, the class of all structurally ordered Abelian groups of finite rank, the class of all structurally ordered free Abelian groups.

We shall call a structurally ordered Abelian group G **atomic** (respectively **atomless**, **with unit**, **without unit**) if the structure $\widehat{G}$ is such. The **width** of a structurally ordered Abelian group G is the maximal number of strictly positive and pairwise orthogonal elements of G, if this number exists and is finite, and the symbol ∞ otherwise. A structurally ordered Abelian group is called a K**-group** if it admits an additional definition of multiplication by real numbers, turning it into a K-space (see (4)). A K-group is the same thing as a complete (in the sense of (5), i.e. conditionally complete as a structure) and divisible structurally ordered Abelian group.

Theorem 3. *Every K-group G decomposes into the direct sum of an atomic and an atomless K-group G_a and G_b. Two K-groups G and G' are elementarily equivalent if and only if G_a and G'_a are elementarily equivalent and G_b and G'_b are elementarily equivalent. All atomless K-groups with unit (respectively without unit) are elementarily equivalent, and such K-groups exist. All atomic K-groups without unit are elementarily equivalent, and such K-groups also exist. Finally, two atomic K-groups with unit are elementarily equivalent if and only if they have one and the same width; moreover, there exist atomic K-groups with unit of any width.*

Theorem 4. *Let M be a class of K-groups and S the set of all natural numbers n such that in M there is a K-group of width n. The class M is decidable if* *and only if the set S is recursive. In particular, the class of all K-groups is decidable.*

Remark. A complete structurally ordered Abelian group G is the direct sum of some K-group G_0 and some complete structurally ordered Abelian group G_1 without divisible elements, see (5). Such G_1 will not be considered here. Let us note only that, contrary to the assertion on p. 139 of the book (5), G_1 is not necessarily a direct sum of linearly ordered cyclic groups. Namely, let Q be an extremally disconnected bicompact space, separable and without isolated points. The existence of such a Q is beyond doubt. Let $C_\infty(Q)$ be the K-group of all continuous and almost everywhere finite (it is permitted to take the values $\pm\infty$ on nowhere dense sets) real functions on Q, see (4). Denote by C^0 the ℓ-subgroup of those functions from $C_\infty(Q)$ all of whose finite values are integers. C^0 is a complete structurally ordered Abelian group without divisible elements, and it is not embedded, with preservation of the operations $+$, $\wedge$, and $\vee$, into a direct sum of linearly ordered Archimedean groups.

In conclusion, about K-lineals. We regard a K-lineal X as a pair $\langle R, X' \rangle$ (a two-sorted model), where R is the field of real numbers, X' is a structurally ordered Abelian group and, at the same time, a vector space over R, and for all $x, y \in X'$ and every positive $\xi \in R$ one has $x < y \to \xi x < \xi y$. By the **width** of a K-lineal X we shall mean the width of the structurally ordered Abelian group X'.

Theorem 5. *If M is a class of K-lineals and the supremum of the widths of the K-lineals from M is equal to ∞, then the class M is undecidable. If n is a natural number and M is the class of all K-lineals of width $\leq n$, then the class M is decidable.*

If X is a K-lineal of width 1, denote by $J(X)$ the linearly ordered set (l.o. set) of jumps of convex subgroups of X'.

Theorem 6. *Two K-lineals X and Y of width 1 are elementarily equivalent if and only if the l.o. sets $J(X)$ and $J(Y)$ are elementarily equivalent. Let M be a class of K-lineals of width 1 and*

$$J(M) = \{J(X) : X \in M\}.$$

The class M is decidable if and only if the class $J(M)$ is decidable. In particular, the class of all K-lineals of width 1 is decidable.

The classification of K-lineals of finite width by elementary properties, with accuracy up to K-lineals of width 1, is analogous to the classification of structurally ordered Abelian groups with a finite number of threads, with accuracy up to linearly ordered Abelian groups.

Theorem 7. Let n be a natural number and let M be the class of all K-linear spaces of width $\leq n$. The class M is decidable.

Theorem 8. Let M be a class of K-spaces. The class M is decidable if and only if the width of the K-spaces in M is bounded in the aggregate by a finite number.

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