ON CHEBYSHEV SETS

L. P. VLASOV

1967

SovietRxiv

Abstract

Full Text

UDC 513.88

MATHEMATICS

L. P. VLASOV

ON CHEBYSHEV SETS*

(Presented by Academician P. S. Novikov on 21 V 1966)

A **Chebyshev set**, according to (^{1}), is a set M in a normed linear space X such that for every $x \in X$ there exists a unique element $x' \in M$ for which $\rho(x, M) = \rho(x, x')$ (abbreviated $xM = xx'$). The point x' is called the **projection** of x onto M. A Chebyshev set M is called **a sun** (^{1}) if, for every $x \notin M$ and for each point of the ray issuing from x' and passing through x, the projection is the point x'. M is called **approximatively compact** (^{2}) if, for every $x \in X$, every sequence $y_n \in M$ with $xy_n \to xM$ has a limit point $y \in M$. The space X is called **locally uniformly convex** (^{3}) if the relations $\|x\| = \|x_n\| = 1$, $\|x + x_n\| \to 2$ imply $x_n \to x$. X is called **uniformly smooth in the direction** $h \in X$ (^{4}) if the limit $\lim_{t\to 0}(\|x + th\| - 1)/t$ exists and is uniform over all x with $\|x\| = 1$.

Lemma 1. *Let M be an approximatively compact Chebyshev set in a normed linear space X. Let $x \notin M$. Then*

$$\lim_{\lambda\to+0} (vM - xM)/vx = 1, \tag{1}$$

where $v = \lambda(x - x') + x$.

Proof. We first show that from $v \to x$ it follows that $v' \to x'$. Assuming the contrary, we find $\varepsilon > 0$ and v_n with $v'_n x' \geq \varepsilon$ for all n. But $xv'_n \to xM$, so that, in view of the approximative compactness of M, v'_n has a limit point $v_0 \in M$. Passing to the limit gives $v_0 x' \geq \varepsilon$, $xv_0 = xM = xx'$, which contradicts the fact that M is a Chebyshev set.

Since $xv' \geq xx'$ and one may assume $xv \leq xx'$, on the segment $[v, v']$ there is a point $\bar{v}$ with $x\bar{v} = xx'$. Note the obvious inequalities

$$v\bar{v} \leq vv' \leq vx'. \tag{2}$$

Put $z = \bar{v} + \dfrac{vx}{xx'}(\bar{v} - x')$. Hence

$$z\bar{v}/\bar{v}x' = vx/xx', \tag{3}$$

and since $v = x + \dfrac{vx}{xx'}(x - x')$, we have

$$z - v = (\bar{v} - x) + \frac{vx}{xx'}(\bar{v} - x),$$

$$vz = (1 + vx/xx')xx' = xx' + vx = vx'. \tag{4}$$

Taking (2), (3), (4) into account, we obtain:

$$0 \leq 1 - (vM - xM)/vx = 1 - (vv' - xx')/vx \leq 1 - (v\bar{v} - xx')/vx$$

$$= (vx + xx' - v\bar{v})/vx = (vx' - v\bar{v})/vx = (vz - v\bar{v})/vx \leq z\bar{v}/vx = \bar{v}x'/xx'.$$

It remains to show that $\bar{v}x' \to 0$. Since $xx' - vx \leq v\bar{v} \leq xx' + vx$, we have $v\bar{v} \to xx'$, and, in view of $vv' \to xx'$, we obtain $\bar{v}v' \to 0$. The required assertion now follows from the inequality $\bar{v}x' \leq v'x' + \bar{v}v'$ and the assertion at the beginning of the proof. The lemma is proved.

Lemma 2. *Let M be an approximately compact Chebyshev set in a Banach space X. Denote by $K(\sigma, x)$ the set*

$$\{z \in X : \; zx \leq \sigma(zM - xM)\}.$$

Then for any $\sigma > 1$ and $x \notin M$ the set $K(\sigma, x)$ intersects every sphere with center at the point x.

Proof. Let $R > 0$ be arbitrary and let V be the ball of radius R with center at x. In the set V one can introduce an order as was done in (5), p. 401. Namely, put $x < x'$ if $x'x \leq \sigma(x'M - xM)$. Antisymmetry and transitivity are obvious. We shall show that V is inductive (see (6), p. 300). Let x_α be a chain in V. For $\alpha < \alpha'$ we have $x_{\alpha'}x_\alpha \leq \sigma(x_{\alpha'}M - x_\alpha M)$. The numerical chain $x_\alpha M$ converges, since it is bounded. Therefore x_α converges in itself, and since X is Banach, $x_\alpha \to x \in V$. It is clear that $xx_\alpha \leq \sigma(xM - x_\alpha M)$, so that $x_\alpha < x$ for every α, and the chain x_α has the upper bound x. This proves that V is inductive. The set $K(\sigma, x) \cap V = \{z \in V : z < x\}$ is also inductive and, by Zorn' s lemma, has a maximal element x_0. Suppose that x_0 lies inside V. Then, by Lemma 1, there is a point $v = \lambda(x_0 - x_0') + x_0 \in V$ with $(vM - x_0 M)/vx_0 \geq 1/\sigma$, which contradicts the maximality of the element x_0. If, however, x_0 lies on the boundary of the ball, then the lemma is proved.

---

* Dedicated to the blessed memory of my friend Mikhail Panteleevich Ulanov.

Theorem 1. *In a Banach space X there do not exist approximately compact Chebyshev sets of the form $M = X \setminus T$, where T is a bounded set.*

This is an immediate consequence of Lemma 2.

Theorem 2. *In a locally uniformly convex Banach space X, every approximately compact Chebyshev set M is a sun.*

Proof. If M is not a sun, then there is a point $x \notin M$ such that not all points of the ray issuing from x' and passing through x have projection x'. It is easy to see that on the ray there exists a point farthest from x' whose projection is x'. Let this point be x. By Lemma 2, for $\sigma_n > 1$, $\sigma_n \to 1$, there is a point x_n with $x_{nx} = xM$, $x_{nx} \leq \sigma_n(x_{nM} - xM)$. Hence $(1 + 1/\sigma_n)xM \leq x_{nM}$, and since $x_{nM} \leq x'_{nx} \leq 2 \cdot xM$, we have $x'_{nx} \to 2 \cdot xM$. Setting $y = (x - x')/xM$, $y_n = (x_n - x)/xM$, we have

$$\|y\| = \|y_n\| = 1, \qquad \|y + y_n\| = x'_{nx}/xM \to 2.$$

Since X is locally uniformly convex, we obtain $y_n \to y$, i.e. $x_n \to 2x - x'$. Obviously, $\rho(2x - x', M) = 2 \cdot xM = \rho(2x - x', x')$. This contradicts the fact that x is the point farthest from x' on the ray with projection x'. The theorem is proved.

Theorem 3. *In a smooth locally uniformly convex Banach space, every approximately compact Chebyshev set is convex.*

It follows from Theorem 2 and the fact that in a smooth normed space every sun is convex (see, for example, (7)).

The assertion of Theorem 3 was given without proof in (11).

Lemma 3. *Let the space X be uniformly smooth in every direction $h \in X$. Then for any $a, b \in X$, $a \neq b$, and $\delta > 0$ there exists $R > 0$ such that for every z with $zc = R$ (where $c = (a + b)/2$) we have*

$$zc + \delta \geq \min\{za, zb\}.$$

Proof. Suppose the contrary, that for some $a, b \in X$, $a \neq b$, and $\delta > 0$, and for every $R > 0$, there is a z with $zc = R$ such that

$$zc + \delta \leq za, \qquad zc + \delta \leq zb. \tag{5}$$

Make a similarity transformation U of the ball of radius $R + \delta$ with center at the point z into the unit ball V: $Ux = (x - z)/(R + \delta)$. If $\mathring{V} = \{x \in V : \|x\| < 1\}$, then, by (5),

$$a' = (a - z)/(R + \delta) \notin \mathring{V}, \qquad b' = (b - z)/(R + \delta) \notin \mathring{V},$$
$$c' = (c - z)/(R + \delta) \in \mathring{V}.$$

Then on the segment $[a', b']$ we find points $a'', b'', a'' \neq b''$, of intersection of the segment $[a', b']$ with the unit sphere. On the segment $[a'', b'']$ some point c'' has the least norm $1-\varepsilon$, whence we have $1-\varepsilon \leq \|c'\| = R/(R+\delta)$, $\varepsilon \geq \delta/(R+\delta)$. Obviously, as $R \to \infty$, $a''b'' \to 0$. Put

$$x = c''/\|c''\| = c''/(1-\varepsilon), \qquad t = b''c''/(1-\varepsilon).$$

Then $t > 0$, $t \to 0$. Let $h = (b-a)/\|b-a\|$. The point c'' was chosen by us so that $\|c'' + \lambda h\| \geq 1-\varepsilon$ for all λ, $-\infty < \lambda < +\infty$; therefore we have $\|x + th\| \geq 1$ and $\|x - th\| \geq 1$. By virtue of uniform smoothness in the direction h, we obtain (x and t depend on R and z):

$$(\|x+th\|-1)/t + (\|x-th\|-1)/t \to 0.$$

Since each summand is positive, $(\|x+th\|-1)/t \to 0$, which is equivalent to the following: $\varepsilon/b''c'' \to 0$. But

$$\delta/ab = \delta/a'b'(R+\delta) \leq \delta/b''c''(R+\delta) \leq \varepsilon/b''c'' \to 0.$$

This is impossible, since a, b, δ do not depend on R. The lemma is proved.

Theorem 4. *In a Banach space that is uniformly smooth in each direction $h \in X$, every approximately compact Chebyshev set is convex.*

Proof. Suppose, to the contrary, that an approximately compact Chebyshev set M is not convex. Then there exist $a, b \in M$, $c = (a+b)/2 \notin M$. Take $0 < \delta < cM$. By Lemma 3 there exists $R > 0$ such that for any z with $zc = R$ we have $zc + \delta \geq \min\{za, zb\} \geq zM$. Take arbitrary $\sigma > 1$. By Lemma 2 there exists z_0 with $R = z_0c \leq \sigma(z_0M - cM)$. We have

$$z_0M - \delta \leq z_0c \leq \sigma(z_0M - cM),$$

whence

$$\sigma \cdot cM \leq (\sigma - 1) \cdot z_0M + \delta \leq (R+\delta)(\sigma-1) + \delta.$$

In view of the arbitrariness of the number $\sigma > 1$, we obtain $cM \leq \delta$, which contradicts the choice of δ. The theorem is proved.

Let us use the occasion to give a series of theorems generalizing the results of note (8) on approximately convex sets. The proofs remain the same.

We shall call a set M **approximately acyclic** if for any $x \in X$ the set $\{y \in M : xy = xM\}$ is acyclic (see (10)).

Theorem 5. *In a Banach space, every boundedly compact and approximately acyclic set is a sun (in the sense of note* (8)*).*

Theorem 6. *In a smooth Banach space, every boundedly compact and approximately acyclic set is convex.*

A special case of this theorem was noted in (3).

Theorem 7. *In an n-dimensional Banach space every approximately acyclic set is convex if and only if the space is smooth.*

The author expresses his gratitude to Yu. A. Shashkin for valuable comments.

Ural State University
named after A. M. Gorky

Received
4 V 1966

References

1. N. V. Efimov, S. B. Stechkin, DAN, **118**, No. 1, 17 (1958).
2. N. V. Efimov, S. B. Stechkin, DAN, **140**, No. 3, 522 (1961).
3. A. R. Lovaglia, Trans. Am. Math. Soc., **78**, 225 (1955).
4. V. Klee, Math. Ann., **142**, 292 (1961).
5. R. R. Phelps, In: Convexity, Proc. of Symp. in Pure Math., **7**, Washington, 1963.
6. N. Bourbaki, *General Topology*, Moscow, 1958.
7. L. P. Vlasov, DAN, **141**, No. 1, 19 (1961).
8. L. P. Vlasov, DAN, **163**, No. 1, 18 (1965).
9. V. Klee, Notices Am. Math. Soc., **12**, No. 7, 812 (1965).
10. S. Eilenberg, D. Montgomery, Am. J. Math., **68**, No. 2, 214 (1946).
11. N. V. Efimov, S. B. Stechkin, Abstracts of Short Communications, ICM, Stockholm, 1962, p. 71.