EXISTENCE OF A CONVEX HYPERSURFACE WITH A GIVEN RELATION BETWEEN CURVATURE FUNCTIONS

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Abstract

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Corresponding Member of the Academy of Sciences of the USSR A. V. POGORELOV

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Let H be a convex regular n-dimensional hypersurface in $(n+1)$-dimensional Euclidean space. Let $R_1, R_2, \ldots, R_n$ be the principal radii of curvature of the hypersurface H, and let $s_1, s_2, \ldots, s_n$ be their elementary symmetric functions. The set function defined on the unit sphere by the equality

$$F_k(\omega) = \int_\omega s_k d\omega,$$

where the integration is over the area of the set ω, is called the k-th curvature function. For $k = n$ it is called the surface function and has a simple geometric meaning. It is the area of that part of the hypersurface H whose spherical image is the set ω. If this property is taken as the definition of the surface function, then it will also make sense for general hypersurfaces without the assumption of regularity.

It is easy to show that if H_λ is a hypersurface equidistant from H, constructed at distance λ from H, then its surface function $F_{\lambda n}$ is expressed in terms of the surface function of H and the curvature functions of H by a formula of the form

$$F_{\lambda n}(\omega) = F_n(\omega) + \lambda c_1 F_{n-1}(\omega) + \ldots + c_n \lambda^n,$$

where the c_k are constant factors independent of the particular hypersurface H chosen. An analogous representation of the surface function of an equidistant hypersurface as a polynomial in powers of λ also holds in the case of a general hypersurface H, without the assumption of regularity. This makes it possible to define the curvature functions for a general hypersurface as the coefficients $F_k(\omega)$ of the expansion of the surface function $F_{\lambda n}(\omega)$ in powers of λ.

Minkowski posed and, under known assumptions, solved the problem of the existence of a closed convex hypersurface with a given surface function. The most general result was obtained by A. D. Aleksandrov, Fenchel, and Jessen

([1]). They proved that for every nonnegative completely additive function $F(\omega)$, given on the unit sphere S, if the conditions

$$\int_S v\,dF = 0, \qquad \int_S |vv_0|\,dF > 0 \tag{1}$$

are satisfied, there exists, and moreover is unique up to a parallel translation, a convex closed hypersurface for which the function F is the surface function. The conditions (1) are also necessary. The first of them expresses the equality to zero of the vector area of the hypersurface, and the second, the positivity of the area of the projection in an arbitrary direction v_0. We shall prove the following general theorem.

Theorem. Let $f_1(v), f_2(v), \dots, f_n(v)$ be arbitrary positive continuous functions of the unit vector v, even in v, i.e. satisfying the condition $f_k(v) = f_k(-v)$. Then there exists a closed convex hypersurface H whose curvature functions $F_k(\omega)$ satisfy the con-

catching

$$F_n(\omega) = \int_\omega f_1(\nu)\,dF_{n-1} + \int_\omega f_2(\nu)\,dF_{n-2} + \dots + \int_\omega f_n(\nu)\,d\omega. \tag{2}$$

If the surface H is regular (twice differentiable), then this means that the elementary symmetric functions S_k of the principal radii of curvature satisfy the equation

$$s_n = f_1 s_{n-1} + f_2 s_{n-2} + \dots + f_n.$$

Proof. First of all, let us note that if the hypersurface H satisfies condition (2), then the homothetically transformed hypersurface with homothety coefficient λ satisfies the condition:

$$F_n(\omega) = \lambda \int_\omega f_1\,dF_{n-1} + \lambda^2 \int_\omega f_2\,dF_{n-2} + \dots + \lambda^n \int_\omega f_n\,d\omega. \tag{3}$$

Thus, it is enough to prove the existence of a hypersurface H satisfying condition (3) for a sufficiently small value of the parameter λ.

Let H' be any centrally symmetric closed convex hypersurface and let $F'_k(\omega)$ be its curvature functions. Put

$$A_\lambda(\omega) = \int_\omega f_1\,dF'_{n-1} + \lambda \int_\omega f_2\,dF'_{n-2} + \dots + \lambda^{n-1} \int_\omega f_n\,d\omega.$$

The set function $\lambda A_\lambda(\omega)$ is nonnegative, completely additive, and satisfies conditions (1) by virtue of the evenness and positivity of the functions f. Consequently, there exists a closed convex surface H_λ for which the function λA_λ will be the surface function. By virtue of the symmetry of the function λA_λ with respect to the center of the unit sphere and the uniqueness of a surface H, this surface also has a center of symmetry. Thus, one obtains a mapping of the set of centrally symmetric convex hypersurfaces into itself. Denote this mapping by B. Then the proof of the existence of a hypersurface satisfying condition (3) reduces to the question of the solvability of the equation

$$H = BH. \tag{4}$$

By virtue of the uniqueness of a hypersurface with a given surface function, the operator B is continuous. Denote by Ω the set of centrally symmetric hypersurfaces that can be placed inside the ball of unit radius. We shall show that, for sufficiently small λ, the operator B maps Ω into itself.

Let

$$M = \max_{\nu,k} f_k(\nu), \qquad m = \min_{\nu,k} f_k(\nu).$$

Then the area of the projection of the surface H_λ, with surface function $F_n(H_\lambda, \omega) = \lambda A_\lambda \omega$, onto any two hyperplanes with normals ν_1 and ν_2 satisfies the inequality

$$m/M \leq \int_S |\nu\nu_1|\, dF_n(H_\lambda, \omega) \Big/ \int_S |\nu\nu_2|\, dF_n(H_\lambda, \omega) \leq M/m. \tag{5}$$

From this inequality one may conclude that the ratio of the radius of the minimal ball containing the hypersurface H_λ to the radius of the maximal ball contained inside the hypersurface H_λ does not exceed a certain number depending only on the numbers M and m, more precisely on their ratio. Since, moreover, the area of the projection of the surface H_λ is small together with λ, for sufficiently small λ the surface H_λ belongs to Ω. Hence, taking into account the compactness of the bounded set of convex hypersurfaces, we conclude that equation (4) is solvable, and consequently that there exists a hypersurface satisfying condition (2). The theorem is proved.

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1. A. D. Aleksandrov, UMN, 5, no. 2, 67 (1947).