ON DETERMINING THE TYPE OF A SPACE ACCORDING TO PETROV' S CLASSIFICATION

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Abstract

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ON DETERMINING THE TYPE OF A SPACE ACCORDING TO PETROV' S CLASSIFICATION

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1. Let us consider a 4-dimensional complex metric space, i.e., a complex differentiable manifold on which a symmetric (metric) tensor is given. Let the components of this tensor in the coordinate system x^i be specified by functions of complex variables $g_{ij}(x)$, which here we shall regard as analytic. The scalar product of vectors, performed with the aid of this metric tensor, is defined without complex conjugation, simply as $A^i B^j g_{ij}$, and may be complex.

At any point of the space under consideration one can compute the Riemann curvature tensor R_{ijkm}, understanding the derivatives entering into it in the sense of the theory of functions of complex variables. Mapping, further, the tensors $g_{ik}g_{jm} - g_{im}g_{jk}$ and R_{ijkm} onto a bivector 6-dimensional complex space, analogously to the way this is done in the Petrov classification of real spaces ([1]), one can determine the system of elementary divisors of the λ-matrix $(R_{ab} - \lambda g_{ab})$, i.e., the type of the original space. Of course, in the formulation of the question considered here, the number of possible types will, generally speaking, be much greater than three.

Since the indicated procedure for determining the type of a space can be carried out at any (nonsingular) point and since, on the other hand, the resulting system of elementary divisors does not depend on the coordinate system used for its determination, the bases of the elementary divisors will be scalar functions of the point and, by virtue of the analyticity of $g_{ij}(x)$, analytic functions of the coordinates.

2. Now suppose that in the original space, which is 8-dimensional in the topological sense, one can distinguish, in the same sense, a 4-dimensional (nonanalytic) subspace such that at each of its points there exist four linearly independent vectors tangent to it, with respect to which the metric tensor g_{ij} is reduced to diagonal form with diagonal $(-1, 1, 1, 1)$ or $(1, -1, -1, -1)$. Because of the 4-dimensionality of the distinguished subspace, any fifth vector tangent to it at some point is a linear combination of the indicated four vectors with real coefficients, and therefore has a real norm. In view of this, such a subspace can be identified with the space-time of general relativity.

To determine the internal metric properties of this subspace, one may use the metric tensor $g_{ij}(x)$ with which the coordinate system x^i, introduced above in the "enveloping" complex space, is endowed. It is necessary only to restrict the range of variation of these coordinates to those, generally speaking, complex values which they assume on the subspace under consideration. If the complex metric tensor $g_{ij}(x)$ satisfies the Einstein equations, for example $R_{ij} = \varkappa g_{ij}$, then the distinguished subspace has the same property, since the validity of the equation $R_{ij} = \varkappa g_{ij}$ does not depend on what values the coordinates take or in what directions the derivatives entering into R_{ij} are taken.

Of interest is the case when, in the original space, it is possible to distinguish several analogous subspaces. We shall say—

show that all of them are generated by one and the same complex four-dimensional metric space. These subspaces may turn out not to be isometric to one another in the usual real sense from the point of view of their intrinsic geometric properties. However, they belong to one and the same type, which is the type of the space that generates them, since the procedure for determining the type of a complex space, carried out at any of its points, does not depend on the selected subspaces. In a somewhat narrowed form, and after a natural generalization to spaces that do not have the signature of general relativity, the latter assertion may be stated as follows.

If a certain real metric space admits a complex coordinate transformation under which the reality of the expression for the interval is preserved, then the space obtained as a result of such a transformation has the same type in the sense of the system of elementary divisors of the aforementioned λ-matrix.

For illustration, consider a four-dimensional complex space with coordinate system t, r, θ, φ, possessing the diagonal metric tensor

$$g_{tt} = | - (r-1)/r, \quad g_{rr} = r/(r-1), \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = r^2 \cos^2\theta.$$

Since for real values of the coordinates the components of the metric tensor take real values, the subspace defined by the equations $\operatorname{Im} t = \operatorname{Im} r = \operatorname{Im}\theta = \operatorname{Im}\varphi = 0$ trivially satisfies the stated requirements. This subspace is the Schwarzschild space-time. However, it is not unique. The equations

$$\operatorname{Re} t = \operatorname{Im} r = \operatorname{Re}\theta = \operatorname{Im}\varphi = 0 \quad \text{or} \quad \operatorname{Im} t = \operatorname{Im} r = \operatorname{Re}\theta = \operatorname{Re}\varphi = 0$$

define two more subspaces generated by the same complex space. They are not isometric to the Schwarzschild one. It is convenient (but by no means necessary) to perform the complex coordinate transformation $t' = it$, $\theta' = i\theta$ in the first case and $\theta' = i\theta, \varphi' = i\varphi$ in the second case. Then the latter subspaces become purely real subspaces, and the expressions for the interval element are written as

$$ds^2 = \frac{r-1}{r}\, dt'^2 + \frac{r}{r-1}\, dr^2 - r^2 d\theta'^2 + r^2 \operatorname{ch}^2 \theta'\, d\varphi^2,$$

$$ds^2 = -\frac{r-1}{r}\, dt^2 + \frac{r}{r-1}\, dr^2 - r^2 d\theta'^2 - r^2 \operatorname{ch}^2 \theta'\, d\varphi'^2.$$

These real spaces are solutions of the Einstein equations for vacuum and, together with the Schwarzschild space, belong to Petrov' s first type.

3. Let us note an important consequence and its practical use. If, by a complex coordinate transformation, one succeeds in making the signature of an Einstein space of general relativity definite, $(++++)$ or $(----)$, then such a space belongs to the first type according to Petrov' s classification, since the λ-matrix of an Einstein space with a definite metric has only simple elementary divisors. With the help of this consequence, for example, the belonging of all static spaces to the first type is established directly, since the substitution $t \to it$ makes their metric definite. An analogous substitution shows that the well-known Taub metric also belongs to the first type (see [2], formula 7.3). However, for example, the space of maximal mobility of the second type (([1]), p. 433) cannot be made definite metric by any transformation.

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CITED LITERATURE

[1] A. Z. Petrov, *Einstein Spaces*, Moscow, 1961.
[2] A. H. Taub, Ann. of Math., **53**, 472 (1951).

Note: Figure translations are in progress. See original paper for figures.