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INFINITE-DIMENSIONAL GEOMETRY OF THE UNIT SPHERE OF A BANACH SPACE

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Abstract

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MATHEMATICS

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INFINITE-DIMENSIONAL GEOMETRY OF THE UNIT SPHERE OF A BANACH SPACE

(Presented by Academician L. V. Kantorovich on 23 XII 1966)

1. Numerical characteristics associated with two-dimensional subspaces (moduli of smoothness ([5]) and convexity ([1]), local modulus of convexity ([2])) have proved useful in various questions of the theory of B-spaces. However, the structure of all possible two-dimensional subspaces completely characterizes only Hilbert space, and for an arbitrary B-space "two-dimensional geometry" does not make it possible to clarify the structure of subspaces and turns out not to be connected with many properties of Banach spaces.

In the present note new characteristics of the unit sphere are introduced and their connection with various properties of the space is indicated (see Theorems 1, 4, 4a, 5, 6). Part of the results (Theorems 2, 7) establishes a connection between the notions introduced.

We shall use the following notation: E^n is a closed subspace in B of defect n; E_n is an n-dimensional subspace; $S(B)$ is the unit sphere of the Banach space B ($S(B) = \{x : \|x\| = 1\}$); $E(\mathfrak{A})$ is the closed linear hull of the set $\mathfrak{A} \subset B$; $B_1 \sim B_2$ means that the spaces B_1 and B_2 are isomorphic. Sequences $\{x_i\}_{i=1}^{\infty} \subset B_1$ and $\{y_i\}_{i=1}^{\infty} \subset B_2$ are called **equivalent** if the operator A, $\{Ax_i = y_i\}_{i=1}^{\infty}$, is an isomorphism of the spaces $E(\{x_i\}_{i=1}^{\infty})$ and $E(\{y_i\}_{i=1}^{\infty})$. The spaces B_1 and B_2 are called ε-**isometric** if there exists an operator A, $AB_1 = B_2$ and $\max(\|A\|, \|A^{-1}\|) \leq 1 + \varepsilon$. Functions $f_1(\varepsilon)$ and $f_2(\varepsilon)$ $(0 < \varepsilon \leq 1)$ are called **equivalent** in two cases: 1) if there exist $C_1 > 0$ and C_2 such that $C_1 \leq f_1(\varepsilon)/f_2(\varepsilon) \leq C_2$ $(0 < \varepsilon \leq 1)$; 2) there exists $\varepsilon_0 > 0$ such that $f_1(\varepsilon) = f_2(\varepsilon) = 0$ for $0 < \varepsilon \leq \varepsilon_0$.

2. δ-characteristics. Let $\|x_0\| = 1$, $x_0 \in B$, and $\varepsilon > 0$. Definitions:

$$\delta\text{a)} \quad \delta_n(\varepsilon; x_0, B) = \inf_{E_n \subset B} \sup_{y \in S(E_n)} \|x_0 + \varepsilon y\| - 1;$$

$$\delta\text{b)} \quad \delta^n(\varepsilon; x_0, B) = \inf_{E^n \subset B} \sup_{y \in S(E^n)} \|x_0 + \varepsilon y\| - 1.$$

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It is clear that

$$\delta^{m_2}(\varepsilon; x, B) \geq \delta^{m_1}(\varepsilon; x, B) \geq \delta_{n_1}(\varepsilon; x, B) \geq \delta_{n_2}(\varepsilon; x, B)$$

for any integers $m_1 > m_2$ and $n_1 > n_2$.

$$\delta\text{c)} \quad \delta_0(\varepsilon; x, B) = \lim_{n\to\infty} \delta_n(\varepsilon; x, B); \quad \delta^0(\varepsilon; x, B) = \lim_{n\to\infty} \delta^n(\varepsilon; x, B).$$

The function $\delta_1(\varepsilon; x, B)$ is equivalent (in ε) to the local modulus of convexity. The function $\delta_n(\varepsilon; x, B)$ may be regarded as the n-dimensional (local) modulus of convexity. In ([3]) (a corollary of Theorem 4), in other terms, the following assertion is noted: if there exist n and ε_0, $\varepsilon_0 < 1$, such that $\delta_n(\varepsilon_0; B) = \inf_{x\in S(B)} \delta_n(\varepsilon_0; x, B) > 0$, then the space B is reflexive. Let us note that the function $\delta_0(\varepsilon; x, B)$ no longer bears analogous responsibility for reflexivity. Thus, $\delta_0(\varepsilon; x, l_1) \equiv \varepsilon$.

3. It is not difficult to see that $\delta^0(\varepsilon; x, c_0) = 0$ for $\varepsilon \leq 1$ and any $x \in S(c_0)$. Any subspace c_0 also possesses this property (see, for example, item 2 of Theorem 2). At the same time one can give an example of a space B_θ,

not a subspace c_0, but for which $\delta^0(\varepsilon; x, B_0) = 0$ for $\varepsilon \leq 1$ ($x \in S(B_0)$). However, the following theorem holds:

Theorem 1. a) If there exists $\varepsilon_0 > 0$ such that $\delta^0(\varepsilon; x, B) = 0$ for $\varepsilon \leq \varepsilon_0$, then there exists a subspace B_1, $B_1 \subset B$, isomorphic to c_0. b) If $B \sim c_0$ and for all $x \in S(B)$ the functions $\delta^0(\varepsilon; x, B)$ are equivalent, then there exists $\varepsilon_0 > 0$ such that $\delta^0(\varepsilon_0; x, B) = 0$.

4. **β-characteristics.** Definitions:

$$\beta\text{a)} \quad \beta_n(\varepsilon; x, B) = \sup_{E_n\subset B} \inf_{y\in S(E_n)} \|x + \varepsilon y\| - 1;$$

$$\beta\text{b)} \quad \beta^n(\varepsilon; x, B) = \sup_{E^n\subset B} \inf_{y\in S(E^n)} \|x + \varepsilon y\| - 1;$$

$$\beta\text{c)} \quad \beta_0(\varepsilon; x, B) = \lim_{n\to\infty} \beta_n(\varepsilon; x, B); \qquad \beta^0(\varepsilon; x, B) = \lim_{n\to\infty} \beta^n(\varepsilon; x, B).$$

Theorem 2. a) For any positive integers m and n:

$$\delta^m(\varepsilon; x, B) \geq \beta_{n+1}(\varepsilon; x, B) \geq \delta_{n+1}(\varepsilon; x, B) \geq \beta^n(\varepsilon; x, B).$$

In particular*

$$\delta^0(\varepsilon; x, B) \geq \beta_0(\varepsilon; x, B) \geq \delta_0(\varepsilon; x, B) \geq \beta^0(\varepsilon; x, B).$$

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b) Let $x \in S(B_1)$, and let B_1 be a subspace of the space B. Then

$$\delta_0(\varepsilon; x, B) \leq \delta_0(\varepsilon; x, B_1) \leq \delta^0(\varepsilon; x, B_1) \leq \delta^0(\varepsilon; x, B);$$

$$\beta^0(\varepsilon; x, B) \leq \beta^0(\varepsilon; x, B_1) \leq \beta_0(\varepsilon; x, B_1) \leq \beta_0(\varepsilon; x, B).$$

c) The functions $\delta_0(\varepsilon; x, B)$, $\delta^0(\varepsilon; x, B)$, $\beta_0(\varepsilon; x, B)$, and $\beta^0(\varepsilon; x, B)$ depend continuously on ε ($\varepsilon \geq 0$) and $x \in S(B)$, and do not change under passage to a subspace B_1 of finite defect (for $x \in S(B_1)$).

5. The most nontrivial point of Theorem 2 is the middle inequality in part a) of the theorem. In its proof one uses Dvoretzky' s theorem (Theorem 2 from ([4])). I shall now give a qualitative formulation of the corollary of this theorem which is used in the proof.

For a set $\mathfrak{A} \subset S(B)$ and $\varkappa > 0$, we shall call the set

$$\mathfrak{A}_\varkappa = \{x : \|x\| = 1 \text{ and } \exists y = y(x),\ y \in \mathfrak{A},\ \text{such that } \|x - y\| \leq \varkappa\}$$

the $\varkappa$-extension of $\mathfrak{A}$.

Theorem 3. H_N is a Euclidean space of dimension N. Denote by $\mathfrak{A}_{n;N}$ a closed set $\mathfrak{A}$ contained in $S(H_N)$ and intersecting every subspace $H_n \subset H_N$. For any n and $\varkappa > 0$ there exists an $N = N(n; \varkappa)$ such that an arbitrary set $\mathfrak{A}_{n;N}$ has the following property: for some

$$k > \frac{c}{n}\sqrt{N}\,\frac{\varkappa}{\ln 1/\varkappa}$$

(c is an absolute positive constant) there exists a subspace $H_k \subset H_N$ for which $S(H_k) \subset \mathfrak{A}_{n;N}$.

Strictly speaking, from Theorem 3 one derives a stronger fact than the indicated inequality (from part a) of Theorem 2), which is at the same time geometrically transparent.

Proposition 1. For any $x_0 \in S(B)$, n, $a > 0$, and $\varkappa > 0$, there exist C and a subspace E_n such that for $y \in S(E_n)$

$$C - \varkappa \leq \|x_0 + ay\| \leq C + \varkappa.$$

6. **Examples of δ- and β-characteristics for some spaces.**

a)

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$$\beta^0(\varepsilon; x, l_p) = \delta^0(\varepsilon; x, l_p) = \sqrt[p]{1+\varepsilon^p} - 1 \sim \varepsilon^p/p \quad (1 \le p < \infty).$$

The characteristics β^0 and δ^0 have the same value for spaces of the form

$$\{B_1 \oplus B_2 \oplus ... \oplus B_n \oplus ...\}_{l_p},$$

where B_n are arbitrary finite-dimensional Banach spaces (although the dimension of B_n may grow), and the orthogonal sum is taken in l_p, i.e.

$$\left\| \sum_{n=1}^{\infty} x_n \right\|^p = \sum_{n=1}^{\infty} \|x_n\|_n^p,$$

where $x_n \in B_n$ and $\|x\|_n$ is the norm in B_n.

b) For $0 \le \varepsilon \le 1$,

$$\beta^0(\varepsilon; x, L[0,1]) = \delta_0(\varepsilon; x, L[0,1]) = 0;$$

$$\delta^0(\varepsilon; x, L[0,1]) = \beta_0(\varepsilon; x, L[0,1]) = \varepsilon.$$

* There is reason to suppose that $\delta^0 \equiv \beta_0$ and $\delta_0 \equiv \beta^0$.

c) Denote by $C^0_{[0,1]}$ the space of continuous functions on $[0,1]$ equal to zero at the point 0. For $0 \ll \varepsilon \ll 1$, $\beta^0(\varepsilon; x, C^0_{[0,1]}) = \delta_0(\varepsilon; x, C^0_{[0,1]}) = 0$; $\delta^0(\varepsilon; x, C^0_{[0,1]}) = \beta_0(\varepsilon; x, C^0_{[0,1]}) = \varepsilon$.

7. Theorem 4. For any $\varkappa > 0$ there exists a basic sequence $\{x_k\}_{k=1}^{\infty} \subset B$, $\|x_k\| = 1$, such that, for $a_1 = 1$ and $|a_k| \le 1$,

$$(1-\varkappa) \prod_{j=2}^{\infty} \left[1 + \beta^0 \left(\frac{|a_j|}{\left\| \sum_{k=1}^{j-1} a_{kx} k \right\|}; \sum_{k=1}^{j-1} a_{kx} k \right) \right] \le$$

$$\le \left\| \sum_{k=1}^{\infty} a_{kx} k \right\| \le (1+\varkappa) \prod_{j=2}^{\infty} \left[1 + \delta^0 \left(\frac{|a_j|}{\left\| \sum_{k=1}^{j-1} a_{kx} k \right\|}; \sum_{k=1}^{j-1} a_{kx} k \right) \right]. \qquad (1)$$

Moreover, in the inequalities (1), instead of $\{x_k\}_1^{\infty}$ one may put any sequence $\{y_k\}_{k=1}^{\infty}$ of the form

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$$\left\{ y_k = \sum_{n=n_k+1}^{n_{k+1}} b_{nx} n \right\}_{k=1}^{\infty}, \qquad \|y_k\| = 1.$$

We shall call a basis $\{x_k\}_1^\infty$ **regular** if every sequence $\{y_k\}_1^\infty$, $\|y_k\| = 1$, of the form

$$\left\{ y_k = \sum_{j=n_k+1}^{n_{k+1}} a_{jx} j \right\}_{k=1}^{\infty}$$

is equivalent to $\{x_k\}_1^\infty$.

Corollary 1. Suppose there exist $C_1 > 0$, $C_2 > 0$, $\varepsilon_0 > 0$, and a function $\psi(\varepsilon)$ such that, for $0 < \varepsilon \le \varepsilon_0$,

$$C_1 \psi(\varepsilon) \le \beta^0(\varepsilon; x, B) \le \delta^0(\varepsilon; x, B) \le C_2 \psi(\varepsilon). \tag{2}$$

Then in B there exists a subspace B_1 with a regular unconditional basis $\{x_k\}_1^\infty$ ($\|x_k\| = 1$); moreover, convergence of the series

$$\sum_{k=1}^{\infty} (a_{kx} k)$$

is equivalent to convergence of the numerical series

$$\sum_{k=1}^{\infty} \psi(a_k),$$

provided only that $a_k \to 0$ ($k \to \infty$).

The same is true also in the case when, instead of (2), for some n one has

$$C_1 \psi(\varepsilon) \le \delta_n(\varepsilon; x, B) \le \delta^0(\varepsilon; x, B) \le C_2 \psi(\varepsilon).$$

Corollary 2. If $\beta^0(\varepsilon; x, B) \ge C\varepsilon$, then there exists $B_1 \subset B$, $B_1 \sim l_1$.

Let us note that in every separable B-space one can introduce an equivalent norm in which $\beta^0(\varepsilon; x, B) > 0$ for $\varepsilon > 0$ and $\forall x \in S(B)$.

Theorem 4a. Suppose $\beta^0(\varepsilon; x, B) > 0$ for $\varepsilon > 0$ and any $x \in S(B)$. For any $1 > \varkappa_1 > 0$ and $1 > \varkappa_2 > 0$ there exists a basic sequence $\{x_k\}_1^\infty \subset B$, $\|x_k\| = 1$, such that, for $a_1 = 1$, $\varkappa_1/2^k \le |a_k| \le 1$, or $a_k = 0$ ($k \ge 2$),

$$\prod_{j=2}^{\infty} \left[1 + (1 - \varkappa_2) \beta^0 \left(\frac{|a_j|}{\left\| \sum_{k=1}^{j-1} a_{kx} k \right\|}; \sum_{k=1}^{j-1} a_{kx} k \right) \right] \le$$

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$$\leq \left\| \sum_{k=1}^{\infty} a_{kx} k \right\| \leq \prod_{j=2}^{\infty} \left[1 + (1+\varkappa_2)\delta^0 \left(\frac{|a_j|}{\left\| \sum_{k=1}^{j-1} a_{kx} k \right\|}; \sum_{k=1}^{j-1} a_{kx} k \right) \right]. \qquad (3)$$

Moreover, in the inequalities (3), instead of $\{x_k\}_1^\infty$ one may put any sequence $\{y_k\}_{k=1}^\infty$ of the form

$$\left\{ y_k = \sum_{n=n_k+1}^{n_{k+1}} b_{nx} n \right\}_{k=1}^{\infty}, \qquad \|y_k\| = 1.$$

Under these conditions the restrictions on $\{a_k\}_1^\infty$ are weakened: $a_1 = 1$, $\chi_1/2^{n_k} \leq |a_k| \leq 1$, or $a_k = 0$ $(k \geq 2)$.

The following proposition indicates the sharpness of the inequalities in Theorem 4a.

Proposition 2. Let the functions $\psi_1(\varepsilon, x)$ and $\psi_2(\varepsilon, x)$, continuous in both variables $x \in S(B)$ and $1 \geq \varepsilon \geq 0$, be such that for the sequence $\{x_k\}_1^\infty$ the assertion of Theorem 4a holds for some χ_1 and χ_2 $(1 > \chi_1 \geq 0,\ 1 > \chi_2 \geq 0)$, and with the functions $\psi_1(\varepsilon, x)$ in place of $\beta^0(\varepsilon; x)$ and $\psi_2(\varepsilon, x)$ in place of $\delta^0(\varepsilon, x)$. Then, for $x \in E(\{x_k\}_1^\infty) = B_1$,

$$(1-\chi_2)\psi_1(\varepsilon, x) \leq \beta^0(\varepsilon; x, B_1); \qquad (1+\chi_2)\psi_2(\varepsilon, x) \geq \delta^0(\varepsilon; x, B_1).$$

8. The results of the present section clarify the connection between the function $\beta^0(\varepsilon; x)$ and weak and strong convergence.

Theorem 5. An element $x_0 \in B$, $\|x_0\| = 1$, has the property that, for every generalized sequence $\{x_\alpha\}_{\alpha \in A}$, $\|x_\alpha\| = 1$, from $x_\alpha \to x_0$ (weakly) it follows that $x_\alpha \to x_0$ (strongly) if and only if

$$\beta^0(\varepsilon; x_0, B) > 0 \quad \text{for } \varepsilon > 0.$$

If B^* is separable, then an analogous assertion holds also for sequences.

Theorem 6. If $\beta^0(\varepsilon; x, B) \geq C\varepsilon$ $(0 \leq \varepsilon \leq 1)$ for some $C > 0$ and for all $x \in S(B)$, then for every sequence $\{x_n\}_{n=1}^\infty$ for which $x_n \to 0$ $(n \to 0)$ it follows that $\|x_n\| \to 0$ $(n \to \infty)$.

9. Denote

$$\hat{\delta}_0(\varepsilon; x, B) = \sup_{B_1} \{\delta_0(\varepsilon; x, B_1) : \ B_1 \subset B,\ x \in B_1 \text{ and } \dim B_1 = \infty\},$$

$$\hat{\delta}^0(\varepsilon; x, B) = \inf_{B_1} \{\delta^0(\varepsilon; x, B_1) : \ B_1 \subset B,\ x \in B_1 \text{ and } \dim B_1 = \infty\}.$$

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The notation $\hat{\beta}_0$ (one should take inf) and $\hat{\beta}^0$ (one should take sup) will be understood analogously.

Theorem 7. For any B and $\chi > 0$ there exists a subspace $B_1 \subset B$ such that, for all $x,\ x \in S(B_1)$, and for all $\varepsilon,\ 0 \le \varepsilon \le 1$,

$$\hat{\delta}_0(\varepsilon; x, B_1) \le (1+\chi)\delta_0(\varepsilon; x, B_1); \qquad \hat{\delta}^0(\varepsilon; x, B_1) \ge (1-\chi)\delta^0(\varepsilon; x, B_1);$$

$$\hat{\beta}_0(\varepsilon; x, B_1) \ge (1-\chi)\beta_0(\varepsilon; x, B_1); \qquad \hat{\beta}^0(\varepsilon; x, B_1) \le (1+\chi)\beta^0(\varepsilon; x, B_1).$$

In connection with Theorem 4, the following question is of interest: for the subspace $B_1(\chi)$ indicated in Theorem 7, does the relation

$$(1+\chi)\hat{\beta}^0(\varepsilon; x, B_1) \ge \hat{\delta}^0(\varepsilon; x, B_1)$$

hold?

A proof of this relation would follow, for example, from the following proposition.

Hypothesis A. Let a uniformly continuous function $\varphi(x)$ $(x \in S(B);\ 0 \le \varphi(x) \le 1)$ satisfy the condition: for every infinite-dimensional subspace $B_1 \subset B$,

$$\inf_{x \in S(B_1)} \varphi(x) = \inf_{x \in S(B)} \varphi(x); \qquad \sup_{x \in S(B_1)} \varphi(x) = \sup_{x \in S(B)} \varphi(x).$$

Then $\varphi(x) \equiv C$.

The following assertion, close to the formulated hypothesis, is a consequence of Theorem 3.

Theorem 8. Let, for a continuous function $\varphi(x)$, $x \in S(B)$, there exist an n such that for every subspace $E_n \subset B$

$$\min_{x \in S(E_n)} \varphi(x) \le \alpha; \qquad \max_{x \in S(E_n)} \varphi(x) \ge \beta.$$

Then $\alpha \ge \beta$.

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