

# UNICELLULARITY AND NONUNICELLULARITY OF WEIGHTED SHIFT OPERATORS

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**Abstract**

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**MATHEMATICS**

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## UNICELLULARITY AND NONUNICELLULARITY OF WEIGHTED SHIFT OPERATORS

*(Presented by Academician V. I. Smirnov on 15 III 1966)*

Let  $\{H_i\}_{i=0}^{\infty}$  be a sequence of Hilbert spaces;  $l^2\{H_i\}_0^{\infty}$  the space of sequences  $X$ ,  $X = \{X_i\}_{i=0}^{\infty}$ ,  $X_i \in H_i$ ,  $i \geq 0$ , with finite norm

$$\|X\| = \left( \sum_0^{\infty} |X_i|^2 \right)^{1/2},$$

where  $|X_i|$  is the norm of the element  $X_i$  in the space  $H_i$ ,  $i = 0, 1, \dots$ . If  $H_0 = H_1 = \dots = H$ , then we write  $l^2(H) = l^2\{H_i\}_0^{\infty}$ . Let  $H_0 \subset H_1 \subset \dots$ . Define in the space  $l^2\{H_i\}_0^{\infty}$  the shift operator  $S$  by the equalities  $SX = Y$ ,  $X \in l^2\{H_i\}_0^{\infty}$ , where  $X = \{X_i\}_0^{\infty}$ ,  $Y = \{Y_i\}_0^{\infty}$  and  $Y_0 = 0$ ,  $Y_i = X_{i-1}$ ,  $i \geq 1$ , and the operator  $\Lambda$  of multiplication by a (bounded) sequence of complex numbers  $\{\lambda_i\}_0^{\infty}$  by the equalities  $\Lambda X = Y$ ,  $X \in l^2\{H_i\}_0^{\infty}$ , where  $X = \{X_i\}_0^{\infty}$ ,  $Y = \{Y_i\}_0^{\infty}$  and  $Y_i = \lambda_i X_i$ ,  $i \geq 0$ . Finally, let  $C^q$  be the  $q$ -dimensional complex space and  $l^2 = l^2(C^1)$ .

Consider in the space  $l^2\{H_i\}_0^{\infty}$  the operator  $T = S\Lambda$  (operators of this kind are naturally called weighted shift operators). Of known interest is a complete description of the invariant subspaces\* of the operators  $S\Lambda$  in the spaces  $l^2\{H_i\}_0^{\infty}$  (see also (6)). Beginning with the work of A. Beurling (2) on the operator  $S$  in the space  $l^2$ , constant attempts have been made in this direction (see, for example, (1, 3-5, 7)). All these works (except (1, 3)) adjoin A. Beurling's investigations. In the present note the study, begun in (1), of invariant subspaces of the operators  $S\Lambda$  in the spaces  $l^2$ ,  $l^2(C^q)$  (and also in "two-sided" sequence spaces, see no. 4) is continued. In particular, it is established that there exist nonunicellular (see the definition below) operators  $S\Lambda$ , acting in  $l^2$ , with a sufficiently complicated structure of invariant subspaces (including ones such that  $\lambda_k \neq 0$ ,  $k \geq 0$ ,  $\lim_k \lambda_k = 0$ ).

## 1. Unicellularity of the operators $S\Lambda$ in the space $l^2$ .

Denote by  $P_k$  ( $k = 0, 1, \dots$ ) the operator acting from  $l^2\{H_i\}_0^\infty$  into  $H_k$  according to the rule  $P_{kX} = X_k$ , if  $X \in l^2\{H_i\}_0^\infty$ ,  $X = \{X_i\}_0^\infty$ . Recall that an operator  $T$ , acting in some Hilbert space  $\mathcal{H}$ , is called **unicellular** if, for any two of its invariant subspaces  $H$  and  $H'$ , at least one of the inclusions  $H \subset H'$ ,  $H' \subset H$  holds. For the operators  $T = S\Lambda$  acting in  $l^2$ , unicellularity means that every  $T$ -invariant subspace coincides with one of the subspaces

$$l_n^2 = \{X : X \in l^2, P_{kX} = 0, \text{ if } k < n\}, \quad n = 0, 1, \dots$$

Denote, as usual, by  $l^p$ ,  $1 \leq p < \infty$ , the space of sequences of complex numbers summable to the power  $p$ .

**Theorem 1.** Consider in the space  $l^2$  an operator  $T = S\Lambda$  such that  $\lambda_i \neq 0$ ,  $i \geq 0$ . Let

$$r_i^{(k)} = |\lambda_i \cdots \lambda_{i+k}|, \quad i = 0, 1, \dots; \quad k = 0, 1, \dots,$$

and suppose that the following conditions are fulfilled:

\* Here and below, subspaces are always assumed to be closed.

- 1) There exists an integer  $k$ ,  $k \geq 0$ , such that  $r_i^{(k)} \downarrow 0$ ,  $i \rightarrow \infty$  (i.e.  $\lim_i r_i^{(k)} = 0$ ,  $r_0^{(k)} \geq r_1^{(k)} \geq \dots$ ).
- 2) There exists an integer  $n$ ,  $n \geq 0$ , such that  $\{r_i^{(n)}\}_{i=0}^\infty \in \bigcup_{p < \infty} l^p$ .

Then the operator  $T$  is unicellular.

**Remark 1.** Conditions 1) and 2) of the theorem mean that the sequence  $\{\lambda_i\}_0^\infty$  splits into a finite number of subsequences of the form  $\{\lambda_{i+kq}\}_{k=0}^\infty$ ,  $i = 0, 1, \dots, q-1$ , in each of which  $|\lambda_{i+kq}|$  is nonincreasing as  $k \rightarrow \infty$ , and, moreover,

$$\{\lambda_{kq}\lambda_{1+kq} \cdots \lambda_{q-1+kq}\}_{k=0}^\infty \in \bigcup_{p < \infty} l^p.$$

**Example.**  $\lambda_{2k} = 1$ ,  $k = 0, 1, \dots$ ;  $\lambda_{2k+1} = 1/(2k+1)$ ,  $k = 0, 1, \dots$

**Remark 2.** Theorem 1 remains valid if the space  $l^2$  is replaced by  $l^p$ ,  $1 < p < \infty$ . If  $p = 1$ , then it suffices to restrict oneself only to requirement 1) in the conditions of Theorem 1 in order to obtain unicellularity of the operator  $T$ .

**2. Invariant subspaces of the operators  $S\Lambda$  in  $l^2(C^q)$ .** Consider the operator  $T = S\Lambda$  in the space  $l^2(H)$ , and let

$$T_1 = \sum_{k=0}^{\infty} \hat{\varphi}_k T^k,$$

where  $\hat{\varphi}_k$ ,  $k = 0, 1, \dots$ , are operators in  $l^2(H)$  such that  $\hat{\varphi}_k(\{X_i\}_0^\infty) = \{\varphi_k X_i\}_0^\infty$ , and  $\varphi_k$  are bounded operators in  $H$ . We shall say of operators  $T_1$  with  $\{\|\hat{\varphi}_k\| \cdot |\lambda_0 \dots \lambda_{k-1}|\}_1^\infty \in l^2$  that they are  $l^2$ -functions of the operator  $T$  (and in this case the series  $\sum_0^\infty \hat{\varphi}_k T^k$  converges in the operator norm), and we shall write  $T_1 = \varphi(T)$ .

Let us introduce one more notation. Let  $Y$  be a subset of the Hilbert space  $\mathcal{H}$ ; let  $\Gamma$  be an operator acting from  $Y$  into  $\mathcal{H}$  such that  $\Gamma Y$  is orthogonal to  $Y$ . Put  $Y \dot{+} \Gamma Y = \{X + \Gamma X : X \in Y\}$ .

**Theorem 2.** Let  $T = S\Lambda$  be an operator acting in  $l^2(C^q)$ , and let the sequence  $\{\lambda_k\}_0^\infty$  satisfy all the conditions of Theorem 1. Let  $M$ ,  $M \subset l^2(C^q)$ , be an invariant subspace of the operator  $T$ :  $TM \subset M$ . Then there exist: an integer  $k$ ,  $k \geq 0$ , subspaces  $B_i$ ,  $i \geq 0$ ,

$$B_0 \subset B_1 \subset \dots \subset B_k = B_{k+1} = \dots, \quad B_{k-1} \neq B_k, \quad B_i \subset C^q, \quad i \geq 0,$$

and an  $l^2$ -function  $\varphi(T)$  of the operator  $T$ , with  $\hat{\varphi}_0 = 0$ , mapping  $l^2\{B_i\}_0^\infty$  into  $l^2\{C^q \ominus B_i\}_0^\infty$ , such that

$$M = l^2\{B_i\}_0^\infty \dot{+} \varphi(T)l^2\{B_i\}_0^\infty. \quad (1)$$

Under the assumptions made, the triple  $k$ ,  $\{B_i\}_0^\infty$ ,  $\varphi(T)$  is uniquely determined by the subspace  $M$ . Conversely, every subspace of the form (1) is invariant with respect to the operator  $T$ .

**Corollary.** Every  $T$ -invariant subspace  $M$ ,  $M \subset l^2(C^q)$ , is generated by no more than  $q$  of its elements (i.e. there are  $X^i$ ,  $1 \leq i \leq q$ ,  $X^i \in M$ , such that the closed linear span of the elements  $T^n X^i$ ,  $n \geq 0$ ,  $1 \leq i \leq q$ , coincides with  $M$ ).

**3. Non-unicellularity of the operators  $S\Lambda$  in  $l^2$ .** Consider again the operators  $T = S\Lambda$  in the space  $l^2$ . We shall now show that conditions of the "regularity" type for the decrease of  $|\lambda_k|$ ,  $k \rightarrow \infty$  (see Theorem 1) are essential for the unicellularity of such operators. In this connection one may note that the very fact of the existence of non-unicellular operators  $S\Lambda$  in the space  $l^2$  such that  $\lim_k \lambda_k = 0$ ,  $\lambda_k \neq 0$ ,  $k \geq 0$ , is important. In this sense the question is settled by the following example:

**Example.** The operator  $T = S\Lambda$  in the space  $l^2$  such that  $\lambda_n = 2^{-n}$ .

if  $n \neq 2^p + 1$ ,  $p \geq 1$ , and  $\lambda_n = 2^{-2(p+1)^2}$ , if  $n = 2^p + 1$ ,  $p \geq 1$ , is not unicellular.

However, a more general assertion also holds.

**Theorem 3.** Consider in the space  $l^2$  the operator  $T = S\Lambda$ ,  $\lambda_k \neq 0$ ;  $|\lambda_k| \geq |\lambda_{k+1}|$ ,  $k = 0, 1, \dots$ . Let  $P = \{n_k\}_{k=1}^\infty$  be a sequence of natural numbers  $n_k$ ,  $k = 1, 2, \dots$ , and  $n_{k+1}/n_k > \varkappa > 1$ ,  $k \geq 1$ . Then there exists a sequence  $\{\mu_k\}_1^\infty$ ,

$\mu_k > 0, \mu_k \downarrow 0, k \rightarrow \infty$ , such that, if  $\tilde{T} = S\tilde{\Lambda}$ , where:  $\tilde{\lambda}_n = \lambda_n$ , if  $n \notin P$ , and  $\tilde{\lambda}_n = \mu_k$ , if  $n = n_k, k = 1, 2, \dots$ , with  $|\tilde{\lambda}_k| \leq \mu_k, k \geq 1$ , then the operator  $\tilde{T}$  is not unicellular.

**4. Operators  $S\Lambda$  in the space  $l_Z^2$ .** Let  $Z$  be the set of all integers and

$$l_Z^p = \{X = \{X_k\}_{k \in Z} : X_k \in C^1, k \in Z; \|X\| = (\sum_k |X_k|^p)^{1/p} < \infty\}.$$

Define in the space  $l_Z^p$  the shift operator  $S$  by the equalities  $SX = Y, X \in l_Z^p$ , where  $X = \{X_k\}_{k \in Z}, Y = \{Y_k\}_{k \in Z}$  and  $Y_k = X_{k-1}, k \in Z$ , and the operator  $\Lambda$  of multiplication by the (bounded) sequence  $\{\lambda_k\}_{k \in Z}$  by the equalities  $\Lambda X = Y, X \in l_Z^p$ , where  $X = \{X_k\}_{k \in Z}, Y = \{Y_k\}_{k \in Z}$  and  $Y_k = \lambda_k X_k, k \in Z$ . If  $\lambda$  is a complex number ( $\lambda \neq 0$ ), put  $\text{sign } \lambda = \lambda/|\lambda|$ , and let  $m_k = \text{sign}(\lambda_0 \dots \lambda_{k-1}), k \geq 1, m_k = \text{sign}(\lambda_{-1} \dots \lambda_{k-1})^{-1}, k \leq 0$  (we assume, of course, that  $\lambda_k \neq 0, k \in Z$ ). Finally, let  $M$  be the operator of multiplication by the sequence  $\{m_k\}_{k \in Z}$ , acting in the space  $l_Z^2$ , and  $P_{WX} = \{Y_k\}_{k \in Z}, Y_k = X_k, k \in W$ , and  $Y_k = 0, k \notin W$ , where  $W$  is a subset of  $Z$ .

**Theorem 4\*.** Consider the operator  $T = S\Lambda$  in the space  $l_Z^2$  and let  $\lambda_0 = 1, \lambda_k \neq 0, k \in Z$ . Then we have:

1) The following assertions are equivalent:

- a)  $T$  has nontrivial invariant subspaces;
  - b)  $Z$  is decomposed into a finite number of arithmetic progressions  $Z_1, \dots, Z_n$ , on each of which the modulus  $|\lambda_p|, p \in Z_i, i = 1, \dots, n$ , is constant;
  - c) there exists a natural number  $n$  such that  $|T|^n = rS^n$ , where  $r > 0$  and  $|T|^n = S^n|\Lambda|, |\Lambda|$  is the operator of multiplication by the sequence  $\{|\lambda_k|\}_{k \in Z}$ .
- 2) Let  $Z_1, \dots, Z_n$  be disjoint arithmetic progressions with difference  $n$  and  $\bigcup_1^n Z_i = Z$ . If  $|\lambda_k| = r_i, k \in Z_i$ , where  $r_i > 0, i = 1, \dots, n$ , then the following assertions are equivalent:

- a) the subspace  $H, H \subset l_Z^2$ , reduces the operator  $T$ ;
- b)  $H = M^{-1}H_0$ , where

$$H_0 = \sum_{i=1}^n \oplus_{P_{Z_i}} H_0 \equiv \sum_{i=1}^n H_i$$

and  $SH_i = H_{i+1}, 1 \leq i \leq n-1, SH_n = H_1, H_1 = \{\{a_{nk}\}_{k \in Z} : \{a_p\}_{p \in Z} \in \widehat{H}\}$ , where  $\widehat{H}$  is a subspace of  $l_Z^2, S\widehat{H} = \widehat{H}$ ;

c)  $H = M^{-1}(\chi_E L^2), E \subset \{\lambda : |\lambda| = 1\}, e^{2\pi i/n} E = E$ . (Here  $L^2$  is the space of measurable functions on the circle  $\{\lambda : |\lambda| = 1\}$ , square-summable, and  $\chi_E L^2 = \{\chi_E f : f \in L^2\}, \chi_E$  is the characteristic function of the set  $E$ . The correspondence of a function to the sequence of its Fourier coefficients is the

natural isomorphism of the spaces  $L^2$  and  $l^2_Z$ ; it is precisely in this sense that the equality in item c) is to be understood.)

The proof of Theorem 4 is based on the following lemma, useful also in some other questions.

**Lemma.** *Let  $T$  be a continuous operator in a Hilbert space  $\mathcal{H}$  such that  $T = VR$ , where  $V$  is isometric and  $R$  is self-adjoint. Suppose, moreover, that the operator  $R$  is one-to-one (or, equivalently,  $R\mathcal{H}$  is dense in  $\mathcal{H}$ ). In order that a subspace*

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\* An analogous assertion can also be formulated for the operators  $S\Lambda$  in the spaces  $l^2\{H_i\}_0^\infty$ , but the scope of the present note does not permit doing this (see (5) for the case  $\lambda_k = 1$ ).

$L, L \subset \mathcal{H}$ , reduces the operator  $T$ , it is necessary and sufficient that it reduce the operators  $V$  and  $R$ .

We now turn to the invariant (but not necessarily reducing) subspaces of the operators  $S\Lambda$  in  $l^2_z$ . It is easy to see that the subspaces

$$l^2_k = \{X : X \in l^2_z, P_{iX} = 0, i < k\}, \quad k \in Z,$$

are invariant with respect to any operator of the form  $T = S\Lambda$ . The unicellularity of such an operator means that the subspaces  $l^2_k$ ,  $k \in Z$ , exhaust all  $T$ -invariant subspaces of  $l^2_z$ . It is easy to give examples of non-unicellular operators  $S\Lambda$  in  $l^2_z$  (operators having reducing subspaces or having at least one eigenvalue).

From the results of the preceding section it also follows easily that the operator  $T = S\Lambda$  is non-unicellular in the space  $l^2_z$ , if at least one of the operators  $T_\pm = S\Lambda_\pm$  is non-unicellular in the space  $l^p$ , where  $\Lambda_+(\Lambda_-)$  is the operator of multiplication by the sequence  $\{\lambda_k\}_{k=0}^\infty$  ( $\{\lambda_{-k}\}_{k=0}^\infty$ , respectively).

However, the general question of the existence of unicellular operators of the form  $T = S\Lambda$  in the space  $l^p_z$  remains open.

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*Note: Figure translations are in progress. See original paper for figures.*

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