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Abstract

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MATHEMATICS

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ON FLAT EXTENSIONS OF RINGS

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1. Let R be an arbitrary associative ring, and let S be some multiplicatively closed system (m.c.s.) of its elements, not containing zero. The ring $R(S)$ is called the **left generalized classical ring of fractions** of the ring R with respect to the m.c.s. S, if the ring R can be mapped into the ring $R(S)$ by means of a homomorphism φ such that: a) the elements of $\varphi(S)$ are invertible in the ring $R(S)$; b) the elements of the ring $R(S)$ have the form $\varphi(s)^{-1}\varphi(r)$, where $s \in S$ and $r \in R$; c) if the ring R is mapped into some ring R' by means of a homomorphism φ' satisfying conditions a) and b), then there exists a homomorphism ψ of the ring $R(S)$ into the ring R' such that $\psi(\varphi(r)) = \varphi'(r)$ for any element $r \in R$ ([1]).
2. Everywhere below, by the term **ring** we shall mean an associative ring with identity, and by the term **module over the ring** R a right unitary R-module.

A module A over a ring R is called R-flat if, for every relation

$$\sum_{i=1}^{n} a_i\mu_i = 0, \tag{1}$$

where $a_i \in A$ and $\mu_i \in R$, there exist elements $b_j \in A$ and $\lambda_{ij} \in R$, $j = 1, \dots, n$, such that

$$\sum_{i=1}^{n} \lambda_{ij}\mu_i = 0, \qquad a_i = \sum_{j=1}^{n} b_j\lambda_{ij} \tag{2}$$

(see ([2]), p. 158).

If R is a commutative ring, then the ring $R(S)$ exists for any m.c.s. S ([3]) and is an R-flat module ([4]), Ch. 2, § 2). In proving the latter assertion one uses the fact that, in the commutative case, the elements of which the kernel of the natural homomorphism $\varphi : R \to R(S)$ consists are known. If R is a ring without zero divisors having a left field of fractions K, then K is an R-flat module ([5]).

The proof uses the possibility of representing the field K as the direct limit of the sequence of R-modules $x^{-1}R$, where $0 \neq x \in R$. In essence, in ([4,5]) the notions of exact sequence and tensor product of modules are used.

We shall show that the left classical ring of fractions $R_c(S)$ of the ring R with respect to the m.c.s. S, if it exists, is an R-flat module (using here only the definition of an R-flat module given above). First let us consider a somewhat more general case:

Theorem 1. *The left generalized classical ring of fractions $R(S)$ of the ring R with respect to the m.c.s. S is a $\varphi(R)$-flat module.*

Proof. Let $\varphi(s_i)^{-1}\varphi(r_i)$ and $\varphi(p_i)$, $i = 1, \dots, n$, be such elements of $R(S)$ and $\varphi(R)$, respectively, that the equality

$$\sum_{i=1}^{n} \varphi(s_i)^{-1}\varphi(r_i)\varphi(p_i) = 0. \tag{3}$$

Any finite number of elements of the ring $R(S)$ can be brought to a common denominator, i.e., there exist elements $s_0 \in S$ and $t_i \in R$, $i = 1, \dots, n$, such that

$$\varphi(s_i)^{-1}\varphi(r_i) = \varphi(s_0)^{-1}\varphi(t_i). \tag{4}$$

In view of the equalities (4), after multiplying equality (3) on the left by $\varphi(s_0)$ we obtain the relation

$$\sum_{i=1}^{n} \varphi(t_i)\varphi(p_i) = 0. \tag{5}$$

The equalities (4) can be written in the form

$$\varphi(s_i)^{-1}\varphi(r_i) = \varphi(s_0)^{-1} \cdot \varphi(t_i) + 1 \cdot 0 + \dots + 1 \cdot 0, \tag{6}$$

where 1 is the identity of the ring $R(S)$.

Now put $a_i = \varphi(s_i)^{-1}\varphi(r_i)$, $\mu_i = \varphi(p_i)$, $b_1 = \varphi(s_0)^{-1}$, $b_2 = \dots = b_n = 1$, $\lambda_{i1} = \varphi(t_i)$, and $\lambda_{ij} = 0$ for $j \neq 1$. In this notation equality (3) gives equality (1), while equalities (5) and (6) give equalities (2). This means precisely that the ring $R(S)$ is a $\varphi(R)$-flat module. The theorem is proved.

Corollary. *If the ring R has a left classical ring of fractions $R_c(S)$ with respect to an m.c.s. S, then the latter is an R-flat module.*

Proof. In this case the m.c.s. S contains no zero divisors of the ring R. Therefore $R(S) = R_c(S)$ and the kernel of the homomorphism φ is zero, i.e. $\varphi(R) = R$ (see ([1])). It remains to apply the theorem.

3. In Richman' s paper ([6]), a superring B in the commutative domain R, contained in its field of fractions K, is called a **generalized ring of fractions** of the ring R if the ring B is an R-flat module. In ([7]) this definition is generalized to the case when R is an arbitrary commutative ring, the ring B is an R-flat module, and $R \subseteq B \subseteq \overline{R}$, where $\overline{R}$ is the complete classical ring of fractions of the ring R. In ([8]) the situation is considered in which a commutative ring R is mapped into a ring B by means of a homomorphism f, and the ring B is contained in the complete classical ring of fractions $\overline{f(R)}$ of the ring $f(R)$, i.e. $f(R) \subseteq B \subseteq \overline{f(R)}$. If the ring B is an R-flat module, then it is also called a **generalized ring of fractions** of the ring R. It is shown that this ring of fractions preserves certain properties inherent in classical rings of fractions. Most of these properties are specific to the commutative case, since in ([6–8]) the questions studied are mainly those connected with the notions of prime and primary ideals and with the notion of closedness. However, there are also properties which it is meaningful to study in the associative case.

Let the ring R have a left generalized classical ring of fractions $R(S)$ with respect to an m.c.s. S, and let B be such a ring that $\varphi(R) \subseteq B \subseteq R(S)$. If the ring B is a $\varphi(R)$-flat module, then we shall call it a **right** $R(S)$**-flat extension** of the ring R. This term seems to us more convenient, since there are many quite different generalized rings of fractions.

4. Let L be a left ideal of the ring R, and $L(S)$ a left ideal of the ring $R(S)$. The **extension of the ideal** L is the left ideal

$$L^e = R(S)\varphi(L)$$

of the ring $R(S)$, and the **contraction of the ideal** $L(S)$ is the left ideal

$$L(S)^c = \varphi^{-1}(L(S) \cap \varphi(R))$$

of the ring R. The ideal L is called **contracted** if $L^{ec} = L$, and the ideal $L(S)$ is called **extended** if $L(S)^{ce} = L(S)$. All left ideals of the ring $R(S)$ are extended, and between the set of all left ideals of the ring $R(S)$ and the set of all contracted left ideals of the ring R one can establish a one-to-one correspondence ([1]). Let us consider right ideals of the ring $R(S)$.

Theorem 2. *If the ring B is a left $R(S)$-flat extension of the ring R, i.e. B is a left $\varphi(R)$-flat module and $\varphi(R) \subseteq B \subseteq R(S)$, then all right ideals of the ring B are extended.*

Proof. Since $B \subset R(S)$, the elements of the ring B have the form $b = \varphi(s)^{-1}\varphi(r)$, where $s \in S$ and $r \in R$. In the ring B the equality holds

$$\varphi(s) \cdot \varphi(s)^{-1}\varphi(r) - \varphi(r) \cdot 1 = 0.$$

Since B is a left $\varphi(R)$-flat module, there exist elements $\lambda_{ij} \in \varphi(R)$, $i, j = 1, 2$, and elements $b_1, b_2 \in B$, such that

$$\varphi(s)\lambda_{11} + \varphi(r)\lambda_{21} = 0, \qquad \varphi(s)\lambda_{12} + \varphi(r)\lambda_{22} = 0, \tag{7}$$

$$\varphi(s)^{-1}\varphi(r) = \lambda_{11}b_1 + \lambda_{12}b_2, \qquad 1 = \lambda_{21}b_1 + \lambda_{22}b_2. \tag{8}$$

For an arbitrary element $b \in B$ put

$$(\varphi(R) : b) = \{\varphi(r) \in \varphi(R) \mid b\varphi(r) \in \varphi(R)\}.$$

From the equalities (7) we obtain that

$$\varphi(s)^{-1}\varphi(r)\lambda_{21} = -\lambda_{11} \in \varphi(R)$$

and

$$\varphi(s)^{-1}\varphi(r)\lambda_{22} = -\lambda_{12} \in \varphi(R),$$

i.e., the elements λ_{21} and λ_{22} belong to

$$(\varphi(R) : b) = (\varphi(R) : \varphi(s)^{-1}\varphi(r)).$$

If b_0 is an arbitrary element of B, then, by virtue of the second of the equalities (8), we have the relation

$$b_0 = \lambda_{21}b_1b_0 + \lambda_{22}b_2b_0 \in (\varphi(R) : b)B.$$

This means that $B \subset (\varphi(R) : b)B$. Since the reverse inclusion is obvious, $B = (\varphi(R) : b)B$.

Let A be an arbitrary right ideal of the ring B, and let $a \in A$. Since $(\varphi(R) : a)B = B$, there exist elements $\varphi(r_i) \in (\varphi(R) : a)$ and $b_i \in B$, $i = 1, \dots, n$, such that

$$\sum_{i=1}^{n} \varphi(r_i)b_i = 1.$$

Then $a\varphi(r_i) \in A \cap \varphi(R)$ and

$$a = \sum_{i=1}^{n} a\varphi(r_i)b_i \in (A \cap \varphi(R))B,$$

i.e. $A \subset (A \cap \varphi(R))B$. Since the reverse inclusion is obvious, $A = (A \cap \varphi(R))B$. This proves the theorem, since the ring $\varphi(R)$ is embedded in the ring B.

Corollary. *If the ring $R(S)$ is a left $\varphi(R)$-flat module, then all right ideals of the ring $R(S)$ are extended, and a one-to-one correspondence can be established between the set of all right ideals of the ring $R(S)$ and the set of all contracted right ideals of the ring R.*

The first assertion is an obvious consequence of the theorem, and the second is proved in the same way as in Theorem 10 of [1].

5. In conclusion we make several remarks.

In the case of the generalized classical ring of fractions, the notion of a complete ring of fractions has no meaning, since (even in the commutative case) there exist rings for which the generalized classical rings of fractions with respect to different maximal multiplicatively closed systems are not isomorphic [9].

The notion of an $R(S)$-flat extension of a ring R is more general than the notion of a generalized ring of fractions, since a generalized ring of fractions of the ring R is obviously also an $R(S)$-flat extension of it, and an $R(S)$-flat extension B of the ring R is its generalized ring of fractions in the commutative case if and only if $(0 : aR)B = B$ for every element a belonging to the kernel of the homomorphism R into B [9].

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Note: Figure translations are in progress. See original paper for figures.