Solution of a completely hyperbolic equation with initial data given on a line of degeneracy

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Abstract

A solution is sought for an equation

$$L_1 L_2 \cdots L_n = f \quad (y > 0),$$

which is strictly hyperbolic in a certain bounded domain D, where

$$L_{iU} \equiv y^{a_i} U_{xx} - U_{yy} + a_i(x,y)U_x + b_i(x,y)U_y + c_i(x,y)U \quad (i = 1, 2, \dots, n),$$

satisfying the initial data

$$\lim_{y \to 0} \frac{\partial^{iU}(x,y)}{\partial y^i} = \tau^{(i)}(x) \quad (i = 0, 1, \dots, 2n-1; \quad a_0 \le x \le b_0).$$

The solution to problem (1), (2) is reduced to n simpler problems. The existence and uniqueness of the solution to the problems under consideration are proven, along with theorems on the well-posedness of their formulation. Bibliography: 7 items.

Full Text

Preamble

In 1967, S. P. Pulkin investigated the boundary value problem for the equation $L_n U = f(x, y, t > 0)$, where the operator is defined as $L_i U = y^{\alpha_i} U_{xx} - U_{yy} + a_i(x,y)U_x + b_i(x,y)U_y + c_i(x,y)U$. Here, $i = 1, 2, \dots, n$, and the domain D is bounded by the interval $[a_0, b_0]$ on the x-axis and the characteristic curves of the operator. The parameters satisfy $0 < \alpha + 2 < \beta < 1$, where $\alpha = \min \alpha_i$.

We consider the singular Cauchy problem for the equation:

$$LU = y^{\alpha} U_{xx} - U_{yy} + a(x,y)U_x + b(x,y)U_y + c(x,y)U = f(x,y) \tag{1}$$

with the initial conditions:

$$\lim_{y\to 0} U(x,y) = \tau_0(x), \quad \lim_{y\to 0} U_y(x,y) = \tau_1(x) \tag{2}$$

for $a_0 < x < b_0$. This problem generalizes results found in [?, ?, ?, ?, ?]. Specifically, it extends the work in [?] for the case $n = 2$ and $L_1 = L_2$. For the system of equations (1)-(2) with $b_1 = b_2$ and conditions (3)-(4) where $\alpha_1 = \alpha_2 = \alpha$, the problem reduces to a second-order system. The existence and uniqueness of the solution are established under specific regularity conditions on the coefficients and initial data.

§ 1. Existence and Uniqueness for the Second-Order Equation

We assume the coefficients of the operator L satisfy the following smoothness conditions in the domain D: $a(x,y) \in C^{r+2}$, $b(x,y) \in C^{r+1}$, and $c(x,y) \in C^r$. Furthermore, we assume the existence of a constant $M < \infty$ such that the coefficients are bounded. The initial functions $\tau_0(x)$ and $\tau_1(x)$ are assumed to be sufficiently smooth on the interval $[a_0, b_0]$.

The solution to the Cauchy problem (1)-(2) can be represented in the form:

$$U(x,y) = \tau_0(x) + y\tau_1(x) + \iint_R \Phi(x,y;\xi,\eta) f(\xi,\eta) d\xi d\eta$$

where R is the characteristic triangle. By substituting this representation into the operator L, we obtain an integral equation for the unknown function $V(x,y)$. Using the method of successive approximations, we demonstrate that the sequence of functions $V_n(x,y)$ converges to a limit $V(x,y)$ that satisfies the boundary conditions:

$$\lim_{y\to 0} V(x,y) = 0, \quad \lim_{y\to 0} V_y(x,y) = 0$$

The operator L can be rewritten in a self-adjoint-like form to facilitate the application of Green' s theorem. Specifically, we consider the identity:

$$[y^\alpha V_x]_x - [y^\alpha V_y]_y = -\{[y^\alpha aV]_x - [y^\alpha bV]_y\} + k(x,y)V + R\phi_N(x,y)$$

where $k(x,y)$ is a function depending on the coefficients a, b, c and their derivatives. By integrating over the domain D and applying the boundary conditions (10), we derive an equivalent Volterra integral equation of the second kind. The kernel of this equation is shown to be integrable, ensuring the existence of a unique solution $V(x,y) \in C^2(D)$.

§ 2. Generalization to Higher-Order Systems

We now extend these results to the n-th order operator $L_n U = f$. Let L_i be defined as in the preamble. We consider the system:

$$L_i W_i = W_{i+1} \quad (i = 1, 2, \dots, n-1), \quad L_n U = W_n \tag{13}$$

with the corresponding initial conditions for each W_i:

$$\lim_{y\to 0} W_i(x,y) = \tau_{i0}(x), \quad \lim_{y\to 0} \frac{\partial W_i}{\partial y} = \tau_{i1}(x) \tag{14}$$

The functions τ_{i0} and τ_{i1} are determined by the higher-order derivatives of the original initial data τ_0, τ_1 and the source term $f(x,y)$.

Under the assumption that $\alpha_k > 4(k-1)$ for $k = 1, \dots, n$, and given sufficient smoothness of the coefficients ($a_i, b_i, c_i \in C^{3n+1}$), we prove that the system (13)-(14) possesses a unique solution. The proof relies on the construction of a sequence of nested integral operators. We show that the solution $U(x,y)$ belongs to the class $C^{2n}(D)$ and satisfies the original equation $L_n L_{n-1} \dots L_1 U = f$ along with the generalized Cauchy conditions.

§ 3. Stability and Convergence

Finally, we address the stability of the solution with respect to the initial data. Let U and $\bar{U}$ be solutions corresponding to initial data (τ_0, τ_1) and $(\bar{\tau}_0, \bar{\tau}_1)$ respectively. We define a norm in the space C^{2n} and demonstrate that:

$$\|U - \bar{U}\|_{C^{2n}} \le K \left(\|\tau_0 - \bar{\tau}_0\|_{C^{3n+1}} + \|\tau_1 - \bar{\tau}_1\|_{C^{3n}}\right)$$

where K is a constant independent of the initial functions. This estimate proves the continuous dependence of the solution on the data.

The iterative process $V_n = TV_{n-1}$ used in the existence proof is shown to be a contraction mapping in a suitably weighted Banach space. The error estimate for the n-th approximation is given by:

$$\rho(V_n, V) \le \frac{M^n}{(n)!} \rho(V_0, V)$$

which ensures rapid convergence. These results provide a complete theoretical framework for solving singular Cauchy problems for a broad class of higher-order hyperbolic equations with degenerating coefficients.

References

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Note: Figure translations are in progress. See original paper for figures.