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THE GROUP OF MOTIONS OF A LINE

MATHEMATICAL PHYSICS

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Abstract

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MATHEMATICAL PHYSICS

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THE GROUP OF MOTIONS OF A LINE AND THE GEOMETRIC THEORY OF FIELDS

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1. As is known, gauge fields A_μ^a are introduced on the basis of the condition of covariance of equations of the Gelfand–Yaglom type ([1]):

$$L^\mu \nabla_\mu \psi + \hat{\lambda}\psi = 0 \tag{1}$$

with respect to localized gauge transformations: $\delta\psi = \varepsilon^a(x) I_a \psi$ (I_a are the generators of a representation of some group).

The covariant derivative ∇_μ is defined by the commutation rule:

$$[\nabla_\mu \delta] = I_a \delta A_\mu^a,$$

where

$$\delta A_\mu^a = \varepsilon^b f_{bc}^a A_\mu^c + \partial_\mu \varepsilon^a. \tag{2}$$

The geometric theory of the potentials A_μ^a ([2,3]), according to which A_μ^a are regarded as coefficients of an affine connection of an internal space, presupposes a certain single-valued process of raising parametric indices, with the generators I_a and I^a related by the relation:

$$[I^a I_b] = f_b^{ac} I_c. \tag{3}$$

2. Relation (3) may be regarded as the structure formula of the given group only in the case when there exists a Casimir operator

$$H = g^{ab} I_a I_b, \quad \text{where } g^{ab} = f_m^{al} f_l^{bm}, \quad \text{and} \quad [I^a I^b] = f_c^{ab} I^c.$$

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From $[HI_a] = 0$ it follows ([4]) that

$$(g^{ab} f^c_{al} + g^{lc} f^b_{al}) I_c I_b = 0,$$

and hence $f^{bc}_a = g^{bl} f^c_{al}$.

But from $f^{bc}_a = g^{bl} f^c_{al}$, obviously, it follows that $f^{ac}_a = 0$. In other words, I_a and I^a form a basis of the algebra of a single structure under the condition that this structure is given by $f_{abc} = f_{[abc]}$, i.e., by constants antisymmetric in all indices. In papers ([2,3]) we essentially restricted ourselves to groups with structure $f_{[abc]}$, i.e., locally compact ones ([4]).

3. Let us admit consideration of an equation of a more general type

$$L^a X_a \psi + \hat{\lambda} \psi = 0, \tag{4}$$

where

$$X_a = \xi^i_a \frac{\partial}{\partial x^i}$$

transforms according to the regular representation of the group

$$\delta X_a = -\varepsilon^b f^c_{ba} X_c.$$

The covariance condition (4) has the form:

$$[L^a_b I] = f^a_{b\,c} L^c. \tag{5}$$

(5), obviously, passes into (3) for groups of structure $f_{[abc]}$.

4. One of the simplest examples of a group of a more general type ($f^a_{b\,c}$ are not antisymmetric in all indices) is the one-dimensional model of the Poincaré group of structure $[X_1 X_2] = -X_2$, isomorphic to the group of motions of the line I (5).

5. Let us investigate the representation I. We find the generators of the regular representation I in differential form. According to (5), introducing the regular representation on the set of left cosets with respect to the Abelian subgroup

$$u(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

we obtain

$$X_\omega f = \left[\frac{df}{dt}(u\omega(t))\right]_{t=0} = \frac{\partial f}{\partial \varepsilon^a} \varepsilon^{a'}(0).$$

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Starting from the fundamental representation I

$$S(\varepsilon,\eta)=\begin{pmatrix} e^{\eta} & 0 \\ \frac{\varepsilon}{\eta}\operatorname{sh}\eta & e^{-\eta}\end{pmatrix},$$

where ε, η are parameters of I; $\eta=\eta_1+\eta_2$;

$$\frac{\varepsilon}{\eta}\operatorname{sh}\eta=e^{\eta_2}\frac{\operatorname{sh}\eta_1}{\eta_1}\varepsilon_1+e^{-\eta_1}\frac{\operatorname{sh}\eta_2}{\eta_2}\varepsilon_2,$$

we obtain

$$X_1=\frac{1}{2}\left[\frac{\partial}{\partial\eta}+\frac{\varepsilon}{\eta}\left(1-\frac{e^{-\eta}\eta}{\operatorname{sh}\eta}\right)\frac{\partial}{\partial\varepsilon}\right];\qquad X_2=\frac{e^{-\eta}}{\operatorname{sh}\eta}\frac{\partial}{\partial\varepsilon}. \tag{6}$$

Then from

$$\frac{\partial\psi_{mn}(\eta)}{\partial\eta}+\frac{m}{\eta}\left(1-\frac{e^{-\eta}\eta}{\operatorname{sh}\eta}\right)\psi_{mn}(\eta)=n\psi_{mn}(\eta),$$

where $f_{mn}(\varepsilon,\eta)=\varepsilon^m\psi_{mn}(\eta)$, it follows that

$$S_{nn'}^{(p,k)}(\varepsilon,\eta)=\sum_{m=0}^{p}\sqrt{\binom{n}{m}}\,\varepsilon^m\eta^{-m}\operatorname{sh}^m\eta\, e^{(2n-m)\eta}\delta_{n-m,n'}; \tag{7}$$

$$0\le m\le p;\qquad k-p\le n\le k;\qquad \psi_{mn}(\eta)\sim\operatorname{sh}^m\eta^{-m}e^{(2n-m)\eta}.$$

In this case

$$X_2 f_{mn}(\varepsilon,\eta)=\sqrt{mn}\, f_{m-1,n-1}(\varepsilon,\eta)$$

(X_2 shifts the columns $S_{nn'}^{(p,k)}$ to the right).

The dimension $s(p,k)$ is specified, as usual, by the character of I:

$$\chi(\eta)=e^{(2k-p)\eta}\frac{\operatorname{sh}(p+1)\eta}{\operatorname{sh}\eta},\qquad s(p,k)=\chi(0)=p+1. \tag{8}$$

From this follows the addition formula for I:

$$T_g f_{mn}(\varepsilon_2\eta_2)=\sum_{k=0}^{m}\sum_{p=0}^{m-k} g_{mn}\binom{m}{k}\binom{m-k}{p}e^{-2p\eta_1}\operatorname{ch}^k\eta\, f_{kn}(\varepsilon'\eta')\, f_{m-k,n-k}(\varepsilon,\eta).$$

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The number k is not an independent weight, since

$$S^{(p,k)}_{nn'} = e^{2(k-p)} S^{(p,p)}_{nn'}.$$

The generators of the representation of weight p

$$M_1 e_n = n e_n, \qquad M_2 e_n = \sqrt{n}\, e_{n-1} \tag{9}$$

follow from the expressions

$$X_1 S^{(p,k)}_{nn'}(\varepsilon, \eta) = n S^{(p,k)}_{nn'}(\varepsilon, \eta),$$

$$X_2 S^{(p,k)}_{n+1,n'}(\varepsilon, \eta) = \sqrt{nm}\, S^{(p,k)}_{n,n'}(\varepsilon, \eta).$$

It is easy to see that $f_{n,m}(\varepsilon, \eta)$ form a basis of the irreducible representation $S^{(p,k)}$.

6. In the particular case, from (7) we obtain the infinite-dimensional representation

$$S^{(\infty)}_{nn'} = \sum_{m=0}^{\infty} \binom{n}{m}^{1/2} \left(\frac{\varepsilon}{\eta} \operatorname{sh} \eta\right)^m e^{(2n-m)\eta} \delta_{n-m,n'}, \tag{10}$$

$$S^{(p,k)}_{nn'} \to S^{(\infty)}_{nn'} \qquad (p \to \infty).$$

From (9) follows (3)

$$S^{(\infty)}_{nn'} = \exp(\eta M_1) \exp\left[\left(\frac{\varepsilon}{\eta} e^{\eta} \operatorname{sh} \eta\right) M_2\right].$$

It turns out that relation (5) is realized only in the case of the infinite-dimensional representation (10).

Indeed, in case I

$$[L^1 M_1] = [L^1 M_2] = 0; \qquad [M_1 L^2] = L^2,$$

$$[L^2 M_2] = L^1; \qquad [M_1 M_2] = -M_2. \tag{11}$$

From Schur’ s lemma ([5,6]) it follows that $L^1 = \lambda E$, but then (11) reduces to Bose statistics:

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$$[A^+A^-] = -E; \qquad M_1 = A^+A; \qquad M_2 = A^-; \qquad L' = -E; \qquad L^2 = A^+, \tag{12}$$

where $A^+ = \sqrt{n+1}\,\delta_{n+1,n'}$, $A^- = \sqrt{n}\,\delta_{n-1,n'}$ $(n, n' = 0, 1, 2, \ldots)$.

Then the covariance conditions (5) for the infinite system of coupled equations are written as

$$(\delta_{nn'}X_1 + \sqrt{n+1}\,\delta_{n+1,n'}X_2)\psi_{n'} + \lambda\psi_n = 0 \qquad (n, n' = 0, 1, 2, \ldots),$$

where $X_1 = \frac{1}{2}(x_1\partial/\partial x_1 - x_2\partial/\partial x_2)$, $X_2 = x_2\partial/\partial x_1$ (x_1, x_2 are the variables of the space of the fundamental representation $I : S(1, \frac{1}{2})$); with respect to I it is realized only on the infinite-dimensional representation $S_a^{(\infty)}$ (12).

7. Let us investigate the invariants I: $\omega = \langle e_\mu \mid L^a \mid e_\nu\rangle$ $(a = 1, 2)$. In the case $S(1, \frac{1}{2})$, the invariant I is given by the convolution $\omega_{\mu\nu} = \langle e_\mu \mid c \mid e_\nu\rangle$, where

$$c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

belongs to the representation $S(1, \frac{1}{2})$ given by (7):

$$S(\varepsilon, \eta) = \begin{pmatrix} e & 0 \\ \dfrac{\varepsilon}{\eta}\operatorname{sh}\eta & e^{-\eta} \end{pmatrix}.$$

Then the invariant convolution is

$$\langle e_\mu \mid S^+cS \mid e_\nu\rangle = \langle e_\mu \mid c \mid e_\nu\rangle.$$

In this sense $S(1, \frac{1}{2})$ are indeed analogous to spinors. In order that invariant convolutions also exist in the case of representations of arbitrary weight $S^{(p,k)}$, let us note that $S^{(p,k)}$ is contained among the irreducible representations into which the direct product of p spinors decomposes:

$$S(1, \tfrac{1}{2}) \times \cdots \times S(1, \tfrac{1}{2}).$$

Thus, in case I the device called the (6) folded direct product is also valid.

It can be shown that $S^{(p+p',\,k+k')}$ is always contained in the decomposition $S^{(p,k)} \times S^{(p',k')}$, and hence, in the general case, the invariant of $S^{(p,k)}$ has the form:

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$$\omega_{\mu\nu} = \left\langle e_\mu \left| \underbrace{c \times \cdots \times c}_{p} \right| e_\nu \right\rangle .$$

8. Let us carry out the decomposition for the product of two conjugate spinors. Obviously, in the general case $S^{(p,k)}$ the formulas obtained extend by complete induction:

$$US(1,1/2) \times [S^{-1}(1,1/2)]^t U^{-1} = U \begin{pmatrix} e^\eta & 0 \\ \frac{\varepsilon}{\eta}\operatorname{sh}\eta & e^{-\eta} \end{pmatrix} \times \begin{pmatrix} e^{-\eta} & -\frac{\varepsilon}{\eta}\operatorname{sh}\eta \\ 0 & e^\eta \end{pmatrix} U^{-1}$$

$$= \begin{vmatrix} e^{2\eta} & 0 & 0 & 0 \\ -\sqrt{2}\,\frac{\varepsilon}{\eta} e^\eta \operatorname{sh}\eta & 1 & 0 & 0 \\ -\varepsilon^2 \frac{\operatorname{sh}^2\eta}{\eta^2} & \sqrt{2} e^{-\eta}\frac{\varepsilon}{\eta}\operatorname{sh}\eta & e^{-2\eta} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} S^{(2,1)} & 0 \\ 0 & f_{00} \end{bmatrix}.$$

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