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Dynamical systems close to Hamiltonian ones

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Date: 1967-01-01T00:00:00+00:00

Abstract

In problems related to determining the number of limit cycles bifurcating from a singular point of the second group, it is sometimes (RZhMat, 1965, 7B199) essential to establish the fact that for a system close to a Hamiltonian one, depending on a parameter μ, under certain additional conditions, the displacement function $\rho(\rho_0, 2\pi, \mu) - \rho_0$ has a zero of order higher than the first with respect to μ at $\mu = 0$. This note demonstrates how this fact can be established for systems of a fairly general form, using the ideas presented in the well-known work of L. S. Pontryagin (ZhETF, 1934, 4, no. 9). Bibliography: 2.

Full Text

Introduction

This section examines the behavior of solutions for the differential equation system discussed in the work of L. S. Pontryagin [?]. We consider the system:

$$\frac{dp}{d\phi} = M(p, \phi, \mu)$$

with the initial condition $r(p, 0) = 0$. Here, the function $r(p, \phi, \mu)$ is defined in the neighborhood $|p - p_0| < \epsilon$ for $0 < \phi < 2\pi$. We define $H(p, \mu)$ as the integral of the function $r(p, \phi, \mu)$ such that $p = p(p_0, \phi, \mu)$ represents the solution starting at $p(p_0, 0, \mu) = p_0$.

Following the methodology established in [?], we analyze the displacement function $p(p_0, 2\pi, \mu) - p_0$. The stability and existence of periodic solutions depend on the derivative of this displacement with respect to the parameter μ. Specifically, we examine the condition:

$$\frac{\partial p(p_0, 2\pi, 0)}{\partial \mu} \neq 0$$

Consider a system of the form:

$$\frac{dx}{dt} = -y + p(x, y, \mu)$$
$$\frac{dy}{dt} = x + q(x, y, \mu)$$

where $p(x, y, 0) = 0$ and $q(x, y, 0) = 0$. We assume the functions p and q are sufficiently smooth in the domain D. In polar coordinates, where $x = \rho \cos \phi$ and $y = \rho \sin \phi$, the system can be transformed to analyze the radial distance ρ. Let $\rho = \rho(\rho_0, \phi, \mu)$ be the solution such that $\rho(\rho_0, 0, \mu) = \rho_0$. The closed trajectory condition for $\mu = 0$ implies that $\rho(\rho_0, 2\pi, 0) = \rho_0$.

To determine the bifurcation of periodic solutions, we evaluate the successor function $h(h_0, \mu)$. For $x > 0$ and $y = 0$, we have $h = H(p, 0)$. The relationship between the displacement in the original coordinates and the transformed system is given by:

$$\frac{dh(h_0, 2\pi, 0)}{d\mu} = H(p_0, 0) \frac{\partial p(p_0, 2\pi, 0)}{\partial \mu}$$

As an application, consider the Hamiltonian $H = -\gamma(x^2 + y^2) + ax^4 + bx^3y + cx^2y^2 - dy^3$. Let the perturbations be defined as:

$$p = \mu(x^3 - 2xy^2)$$
$$q = \mu(2x^2y - y^3)$$

By applying the criteria for the existence of limit cycles as developed by N. N. Bautin [**?**], we can determine the conditions under which the equilibrium point at the origin loses stability and generates a periodic orbit. The results obtained here align with the qualitative theory of differential equations as presented in the cited literature.

References

1. Pontryagin, L. S. (1934). On the dynamical systems close to Hamiltonian systems. *Zh. Eksp. Teor. Fiz.*, 4(9), 1-3.
2. Bautin, N. N. (1965). On the number of limit cycles appearing with the variation of coefficients from an equilibrium state of focus or center type. *Mat. Sb.*, 1(53-66).

Figures

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DIFFERENTIAL EQUATIONS

DECEMBER 1967, VOL. III, NO. 12

SHORT COMMUNICATIONS

UDC 517.917

ON DYNAMICAL SYSTEMS CLOSE TO HAMILTONIAN

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In questions related to finding the number of limit cycles generated from a singular point of the second group, an important role is sometimes played by the establishment of the fact that for a system close to a hamiltonian one, depending ot the parameter μ, under certain additional conditions the succession function $\rho(\rho_0, 2\pi, \mu) - \rho_0$ has at $\mu = 0$ a zero of an order higher than the first with respect to μ.

In this note we will show, how it is possible to establish this fact for a system of sufficienty general type, using the ideas outlined in the well-known work of L. S. Pontryagin [1].

Lemma. Let

$$\frac{d\rho}{d\varphi} = r(\rho, \varphi, \mu) \tag{1}$$

— be a differential equation, defined in polar coordinates; $r(\rho, \varphi, 0) = 0$. The function $r(\rho, \varphi, \mu)$ is assumed to be periodic in φ with period 2π and continuous together with $r_\mu, r_\rho, r_{\mu\rho}$ in the region

$$|\rho - \rho_0| < \varepsilon, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |\mu| < \varepsilon.$$

Let

$$R(\rho, \mu) = \int_0^{2\pi} r(\rho, \varphi, \mu)\, d\varphi,$$

and $\rho = \rho(\rho_0, \varphi, \mu)$ — solution of equation (1), satisfying the initial condition $\rho(\rho_0, 0, \mu) = \rho_0$.

Then

$$\frac{\partial \rho(\rho_0, 2\pi, 0)}{\partial \mu} = \frac{\partial R(\rho_0, 0)}{\partial \mu}.$$

This equerlity is contained in the proof of lemma 2 in [1].

Theorem. Let there be given a system of equations

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y} + p(x, y, \mu), \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x} + q(x, y, \mu), \tag{2}$$

in which $p(x, y, 0) = 0$, $q(x, y, 0) = 0$.

Let C_{h_0} — be a closed curve, the points of which satisfy the equation $H(x, y) = h_0$, and in a certain neighborhood of it and in a finite region D_{h_0}, bronded by it, for sufficiently small $|\mu|$ the functions H, p and q have continuous partial derivatives up to the second order inclusive, with along C_{h_0}

$$xH'_x + yH'_y \neq 0. \tag{3}$$

Let $\rho = \rho(\rho_0, \varphi, \mu)$ — be the equation in polar coordinates of the integral curve of system (2), passing through the point $\varphi = 0$, $\rho = \rho_0$, lying on the curve C_{h_0}.

For the condition $\frac{\partial \rho(\rho_0, 2\pi, 0)}{\partial \mu} = 0$, it is necessary and sufficient that

$$\iint_{D_{h_0}} \left(\frac{\partial^2 p(x, y, 0)}{\partial x \partial \mu} + \frac{\partial^2 q(x, y, 0)}{\partial y \partial \mu} \right) dxdy = 0. \tag{4}$$

Figure 1: Figure 1

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Note that condition (3) guarantees not only the presence of a tangent at each point of the curve C_{b0}, but also the representability of the integral cquation in the form $\rho =$ form $\rho = \rho(\rho_0, \varphi, \mu)$ for small $[\mu]$ and all $\varphi \in [0, 2\pi]$, so that the integral curve curve cannot touch the half-lines $\varphi = \text{const}$.

In connection with this, the proof of the theorem is carried out literally, as in [1], by introducing of C_{ko}, localiear cocstitate csosystem (φ, h), but only one in this cayae, as φ it is posssble to take both the polars sygle, and it may to lernna 2 from [1] neofiagume ρ with an indication of the above-mentioned lerma. Then, in order that $\frac{\partial h(\rho_0, 2\pi, 0)}{\partial \mu} = 0$, is necessary and suctatbve to fulfilll the condition (4). Spece $h = h(h_0, \varphi, \mu)$ — ypabnenine survoi $\rho = \rho(\rho_0, \varphi, \mu)$ in localshold custeme equation.

Since for $x > 0$ and $y = 0$ we have $H = H(\rho, 0)$, then

$$h(\rho_0, 2\pi, \mu) = H(\rho(\rho_0, 2\pi, \mu), 0),$$

whence, taking into account that $\rho(\rho_0, 2\pi, 0) = \rho_0$, we obtain

$$\frac{\partial h(\rho_0, 2\pi, 0)}{\partial \mu} = \frac{\partial H(\rho_0, 0)}{\partial x} \frac{\partial \rho(\rho_0, 2\pi, 0)}{\partial \mu}.$$

Since by virtue of $\frac{\partial H(\rho_0, 0)}{\partial x} \neq 0$, the assertion of the theorem

Example. $H = \frac{1}{2}(x^3 + y^2) + ax^4 + bx^3y + cx^2y^2 - bxy^3 + ay^4$;

$$p = \mu(x^3 - 2sy^2); \quad q = \mu(2x^2y - y^3).$$

Condition (4) is fulfilled here for all sufficiently small h_0, since upon replacing x with $-y$, and y for x is

$$\iint_{D_{h_0}} (x^2 - y^2)\, dxdy$$

the integrand changes sign, while the equation $H = h_0$ does not change. Condition (3) is fulfilled here for all sufficiently small h_0. Then $\frac{\partial p(\rho_0, 2\pi, 0)}{\partial \mu} = 0$ for all sufficiently small ρ_0.

This result, established in this example, was used also in the proof in [2] in docasing causing that the maximum number of limit cycles, arising from a singular point of an autonomous system with purely imaginary eigenvalues of the linear part and homogeneous polynomial nonlinearities of the third degree, is equal to five.

Using this opportunity, I express to N. N. Bautin a fair remark about an inaccuracy in the footnote on p. 60 in [2].

Literature

1. Pontryagin L. S. JETP, 4, v. 9, 1—3, 1934.
2. Sibirŗskiý K. S. Differential Equations, 1, No. 1, 53—66, 1965.

Received by the editors 10 November 1963.

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Figure 2: Figure 2