REGULARIZED SUMS OF ROOTS

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Abstract

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MATHEMATICS

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REGULARIZED SUMS OF ROOTS OF A CLASS OF ENTIRE FUNCTIONS

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1°. Consider an entire function $f(z)$ of the form

$$f(z) = \sum_{k=0}^{N-1} e^{a_k z} P_k(z). \tag{1}$$

Here a_k are complex constants; $P_k(z)$, as $z \to \infty$, are asymptotic series

$$P_k(z) \sim \sum_{\nu=-n_k}^{\infty} \frac{c_\nu^{(k)}}{z^\nu}, \qquad c_{-n_k}^{(k)} \neq 0. \tag{2}$$

It is assumed that the z-plane can be divided into a finite number of sectors in each of which the $P_k(z)$ are analytic functions for $|z| > R$, and the representation (2) admits differentiation.

We shall agree to call the entire functions (1) **functions of class** K. The numbers a_k and $c_\nu^{(k)}$ in (1) and (2) will be called the **parameters of the asymptotics** of $f(z)$.

Functions of class K arise in solving boundary-value problems containing a parameter z. Consider, for example, on the interval $0 \leq x \leq 1$ the differential equation

$$d^n y/dx^n + a_1(x,z) d^{n-1}y/dx^{n-1} + \cdots + a_n(x,z) y = 0, \tag{3}$$

where

$$a_q(x,z) = z^q \sum_{j=0}^{q} z^{-j} p_{ij}(x) \qquad (q = 1, 2, \ldots, n).$$

Let $a_{q0}(x) = a_{q0} r(x)$ $(q = 1, 2, \dots, n)$, where $r(x) > 0$, while a_{q0} are complex constants, and suppose that the equation
$\lambda^n + a_{10}\lambda^{n-1} + a_{20}\lambda^{n-2} + \dots + a_{n0} = 0$ has distinct roots. Then, under the condition of infinite differentiability of the coefficients $a_{qj}(x)$, the eigenvalues of equation (3) with boundary conditions polynomially depending on z are roots of a function $f(z) \in K$. Moreover, the asymptotic parameters of $f(z)$ are expressed explicitly in terms of the coefficients of equation (3) and of the boundary conditions. This fact follows from the work of J. D. Tamarkin ([1]). In the case where the coefficients of the equation are p times differentiable, the representation (1) holds with the sole difference that

$$P_k(z) \sim \sum_{\nu=-n_k}^{p} \frac{c_\nu^{(k)}}{z^\nu}. \tag{2'}$$

In the present work we find the values of the regularized sums of the roots of an arbitrary function $f(z) \in K$. The values of the regularized sums of the eigenvalues of the Sturm–Liouville equation were found in

fundamental works in this area by I. M. Gel' fand, B. M. Levitan and L. A. Dikii (see ([2–4])). The theorems given below are of a function-theoretic character and are not connected with differential operators. However, they make it possible to find, by a unified method, regularized sums of eigenvalues for differential equations of arbitrary orders.

2°. Let $f(z) \in K$ (see (1)). Mark on the complex z-plane the numbers

$$\overline{\alpha_k} \qquad (k = 0, 1, 2, \dots, N-1) \tag{4}$$

and denote their convex hull by P (see Fig. 1). In the general case P is an r-gon: $r \le N$. We shall call the directions of the exterior normals to P critical. Remove from the z-plane r sectors T_s $(s = 0, 1, 2, \dots, r-1)$ of arbitrarily small aperture with bisectors parallel to the critical directions and with common vertex at zero. Denote the remaining domain by Ω. It, in turn, decomposes into r open sectors Ω_s $(s = 0, 1, \dots, r-1)$. It can be shown that in the intersection of the domains Ω and $|z| > R$, for sufficiently large R, there are no zeros of $f(z)$. Choose in one of the sectors Ω_s (for definiteness Ω_0) a contour Γ_0 as indicated in Fig. 1. Obviously, one may assume that $f(z) \ne 0$ for $z \in \Gamma_0$. Assuming, moreover, that $f(0) \ne 0$, introduce into consideration the integral

$$Z_0(\sigma) = \frac{1}{2\pi i} \int_{\Gamma_0} z^{-\sigma} \frac{f'(z)}{f(z)}\, dz, \tag{5}$$

Fig. 1

which we shall call the **zeta-function associated with** $f(z)$*. It is not difficult to show that, as $z \to \infty$, $z \in \Gamma_0$,

Fig. 1

Figure 1: Fig. 1

$$\frac{f'(z)}{f(z)} \sim \sum_{\nu=0}^{\infty} \frac{\omega_\nu^{(0)}}{z^\nu}, \tag{6}$$

where ω_ν^0 are expressed in terms of the asymptotic parameters of $f(z)$. In view of (6), $Z_0(\sigma)$ is regular for $\operatorname{Re}\sigma > 1$, if one also takes into account that

$$\int_{\Gamma_0} z^{-\sigma} \sum_{\nu=0}^{q} \frac{\omega_\nu^{(0)}}{z^\nu}\, dz \equiv 0, \qquad q \geq 0 \quad \text{for } \operatorname{Re}\sigma > 1,$$

then it is not difficult to arrive at the following assertions:

Lemma 1. *The zeta-function $Z_0(\sigma)$ associated with a function $f(z)$ of the class K admits analytic continuation to the entire complex σ-plane as an entire function**.*

* In (5), $z^{-\sigma} = e^{-\sigma \operatorname{Ln} z}$, where $\operatorname{Ln} z$ is a fixed regular branch of the logarithm outside Γ_0.

** Under condition (2′) one may assert that $Z_0(\sigma)$ continues to the half-plane $\operatorname{Re}\sigma > -p - n_0 - 1$.

Lemma 2. *For* $\operatorname{Re}\sigma > 1$ *the representation*

$$Z_0(\sigma) = \sum_{(k)} z_k^{-\sigma}, \tag{7}$$

holds, where z_k are the roots of $f(z)$.

Lemma 3. *The equalities*

$$Z_0(-m) = \omega_{m+1}^{(0)!} \qquad (m = 0, 1, ...), \tag{8}$$

are valid, where $\omega_\nu^{(0)}$ are the coefficients in the expansion (6).

3°. We shall now assume, for simplicity, that on the boundary P there lie only those numbers (4) which fall at the vertices of P. The remaining $N-r$ numbers are thus situated inside P. In this case, for the roots of $f(z)$ the following asymptotic formula* is valid as $n \to \infty$:

$$z_{n,s} \sim a_s n \left\{ 1 + \frac{b_s \ln n + c_s}{n} + \sum_{k=1}^{\infty} \frac{R_k^{(s)}(\ln n)}{n^{k+1}} \right\}. \tag{9}$$

(see [5,6]). Here s is the number of the sector T_s in which the series of roots lies, and n is the number of the root determined by the asymptotic formula (9). $R_k^{(s)}(u)$ are polynomials of degree k in u. All coefficients in (9) are determined by the asymptotic parameters of $f(z)$. From (9) we easily obtain

$$z_{n,s}^{-\sigma} \sim \sum_{k=0}^{\infty} \frac{Q_k^{(s)}(\sigma, \ln n)}{n^{k+\sigma}}, \tag{10}$$

where

$$Q_k^{(s)} = a_s^{-\sigma} \sum_{\nu=0}^{k} d_{k,\nu}^{(s)}(\sigma) \ln^{\nu} n,$$

and $d_{k,\nu}^{(s)}(\sigma)$ are polynomials in σ. On the basis of (10) one can conclude that the series**

$$\psi_m(\sigma) = \sum_{n=1}^{\infty}{}' \sum_{s=1}^{r} \left[z_{n,s}^{-\sigma} - \sum_{k=0}^{m+1} \frac{Q_k^{(s)}(\sigma, \ln n)}{n^{k+\sigma}} \right] \tag{11}$$

admits analytic continuation to the half-plane $\operatorname{Re}\sigma > -m-1$. The numbers $\psi_m(-m)$ will be called the **regularized m-sums of the roots** of $f(z)$. The latter can be expressed in terms of the asymptotic parameters of $f(z)$.

Indeed, write (11) in the form

$$\psi_m(\sigma) = Z_0(\sigma) - \Phi_m^{(0)}(\sigma),$$

where

$$\Phi_m^{(0)}(\sigma) = \sum_{n=1}^{\infty}{}' \sum_{s=1}^{r} \left(\sum_{k=0}^{m+1} \frac{Q_k^{(s)}(\sigma, \ln n)}{n^{k+\sigma}} \right) =$$

$$= \sum_{k=0}^{m+1} \sum_{\nu=1}^{k} a_s^{-\sigma} \left(\sum_{s=1}^{r} d_{k,\nu}^{(s)}(\sigma) \right) \sum_{n=1}^{\infty} \frac{\ln^{\nu} n}{n^{k+\sigma}} = \sum_{k=0}^{m+1} \sum_{\nu=0}^{k} a_s^{-\sigma} d_{k,\nu}(\sigma) (-1)^{\nu} \frac{d^{\nu}}{d\sigma^{\nu}} \zeta(k+\sigma). \tag{12}$$

Here $\zeta(z)$ denotes the Riemann zeta-function. Representation (12) makes it possible to find the value $\Phi_m^{(0)}(-m)$ for $m = 0, 1, 2, \ldots$. Together with (8), this allows us to establish the following theorem:

Theorem 1. *For any* $m = 0, 1, 2, \ldots$ *the equalities*

$$\sum_{n=1}^{\infty}{}' \sum_{s=1}^{r} \left(z_{n,s}^{m} - \sum_{k=0}^{m+1} \frac{Q_k^{(s)}(-m, \ln n)}{n^{k-m}} \right) = \omega_{m+1}^{(0)!} - \Phi_m^{(0)}(-m) \tag{13}$$

are valid.

Let us make several remarks in connection with Theorem 1:

a) It is easy to verify that $O_1^{(s)}(0, \ln n) = 1$, $O_1^{(s)}(0, \ln n) = 0$. If, therefore, in the left-hand sides of (13) we put $m = 0$, then we obtain an integer equal to the excess or deficiency of roots under the method of enumeration chosen by us, in accordance with the asymptotic formula (9). We shall call this number the **regularization defect** and denote it by $\varkappa$. According to (13)*

$$\varkappa = \omega_1^{(0)} - \Phi_0^{(0)}(0) = \omega_1^{(0)} + \frac{r}{2} - \sum_{s=1}^{r} b_s \operatorname{Ln} a_s + \sum_{s=1}^{r} c_s. \tag{14}$$

b) Since the remainders of the series (13) are easily estimated, formulas (13) can be used to write a system of algebraic equations that determines the roots of $f(z)$ lying in a finite circle.

c) Using the independence of the right-hand sides of (13) from the choice of the contour Γ_0, one can obtain a linear recurrent system relating the coefficients of the asymptotic expansions (9) to the parameters of the asymptotics of $f(z)$. If, moreover, one takes into account the relations arising as a result of equating to zero the coefficients at the poles of

$$\frac{d^{\nu}}{d\sigma^{\nu}} \zeta(k + \sigma)$$

of the right-hand side of (12), then a recurrent system of equations arises which determines all coefficients of the asymptotic expansions (9).

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* An analogous asymptotic formula is valid in the case when the numbers (4) divide the sides of P into commensurable parts.

** Since the first index of the root $z_{n,s}$ is determined by the asymptotic formula, the finite number of roots may in this case turn out to be unnumbered or, conversely, may prove to be indexed several times over. The prime over the summation sign means that in the first case the unnumbered roots are included in the sum, while in the second the redundant first terms in the square brackets are replaced by zeros.

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* In the right-hand side of (14) one takes the values of the logarithmic function defined in the footnote to formula (5).

Note: Figure translations are in progress. See original paper for figures.