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AERODYNAMICS

1967

SovietRxiv

Abstract

Full Text

UDC 533.13.16+533.6.5

AERODYNAMICS

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DAMPING OF DISTURBANCES INTRODUCED BY A BODY OF REVOLUTION INTO A SUPERSONIC FLOW OF A VISCOUS HEAT-CONDUCTING GAS

(Presented by Academician A. A. Dorodnitsyn, 9 IX 1966)

Let us consider the flow past a body of revolution by a stream of a real gas, in which dissipative processes associated with viscosity and heat conduction may occur. To analyze the asymptotic pattern of the flow far from the body, we shall use the equations of continuity, Navier–Stokes, and heat transfer ([1]):

$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_r}{\partial r} + \frac{\rho v_r}{r} = 0; \tag{1}$$

$$\rho\left(v_x\frac{\partial v_x}{\partial x} + v_r\frac{\partial v_x}{\partial r}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\lambda_1\frac{\partial v_x}{\partial x} + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)\right] + \frac{\partial}{\partial r}\left[\lambda_1\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)\right] + \frac{\lambda_1}{r}\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right); \tag{2}$$

$$\rho\left(v_x\frac{\partial v_r}{\partial x} + v_r\frac{\partial v_r}{\partial r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial x}\left[\lambda_1\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)\right] + \frac{\partial}{\partial r}\left[2\lambda_1\frac{\partial v_r}{\partial r} + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)\right] + \frac{2\lambda_1}{r}\left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r}\right); \tag{3}$$

$$\rho T\left(v_x\frac{\partial s}{\partial x}+v_r\frac{\partial s}{\partial r}\right)=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial r}\left(k\frac{\partial T}{\partial r}\right)+\frac{k}{r}\frac{\partial T}{\partial r}$$
$$+2\lambda_1\left[\left(\frac{\partial v_x}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial v_x}{\partial r}+\frac{\partial v_r}{\partial x}\right)^2+\left(\frac{\partial v_r}{\partial r}\right)^2+\left(\frac{v_r}{r}\right)^2\right]$$
$$+\left(\lambda_2-\frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\right)^2. \tag{4}$$

Here x and r are cylindrical coordinates; v_x and v_r are the components of the velocity vector along the axes x and r; ρ is density; p is pressure; s is specific entropy; T is temperature; λ_1 is the viscosity coefficient; λ_2 is the second viscosity coefficient; k is the coefficient of thermal conductivity.

We close the written system with the aid of two differential relations

$$ds=\frac{c_p}{\alpha\rho a^2 T}\left(dp-a^2 d\rho\right),\qquad dT=\frac{1}{\alpha\rho a^2}\left(\chi\, dp-a^2 d\rho\right), \tag{5}$$

$$\left(\alpha=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p,\qquad a^2=\left(\frac{\partial p}{\partial\rho}\right)_s,\qquad \chi=\frac{c_p}{c_v}\right),$$

which connect the thermodynamic functions ρ, p, s, and T with one another. In formulas (5), α is the coefficient of thermal expansion, a is the adiabatic speed of sound, and χ is the ratio of the specific heat at constant pressure c_p to the specific heat at constant volume c_v.

We introduce a new system of coordinates n, τ, associated with the characteristics of the undisturbed flow, which are obtained when the Navier–Stokes and heat-transfer equations are replaced by the Euler equations and by the equation of conservation of entropy in a particle. We denote by A_∞ and M_∞, respectively, the angle and the Mach number

Mach number of the incident uniform flow; then

$$x=n\sin A_\infty+\tau\cos A_\infty,\qquad r=-n\cos A_\infty+\tau\sin A_\infty$$
$$(\sin A_\infty=1/M_\infty,\quad M_\infty=v_\infty/a_\infty). \tag{6}$$

Analogous formulas are valid for the components v_n, v_τ of the velocity vector v along the axes n, τ. In characteristic variables, instead of equations (2) and (3), it is convenient to consider their linear combinations, which are the n- and τ-components of the Navier–Stokes equation.

In analyzing the system of equations (1)–(5), referred to the new coordinates, we shall assume that the values of all gas parameters in the region of space under consideration deviate only slightly from the corresponding values in the incident uniform flow. With respect to the velocity of the latter, we shall assume that its magnitude is greater than the speed of sound and that it is directed along the x-axis* . We shall also assume that, far from the body being streamlined, the disturbances are concentrated in a narrow region elongated along the characteristics of the initially uniform flow. The width of the disturbed region must be much smaller than the distance to the body. In accordance with the remarks made, we pass to dimensionless variables:

$$n = Ln', \qquad \tau = \frac{L}{\Delta}\tau', \qquad v_n = a_\infty(1 + \varepsilon v_n'), \qquad v_\tau = a_\infty(\operatorname{ctg} A_\infty + \varepsilon \Delta v_\tau'),$$

$$\rho = \rho_\infty(1 + \varepsilon\rho'), \qquad p = p_\infty(1 + \varepsilon p'), \qquad a = a_\infty(1 + \varepsilon a'). \tag{7}$$

Here L is the characteristic length in the direction of the n-axis; ε and Δ are small numerical parameters, and the index ∞ refers to the gas parameters in the incident flow.

As a result of substituting formulas (6) and (7) into the original system of equations (1)–(5), three dimensionless coefficients are obtained: the Reynolds numbers $\mathrm{Re}_1 = \rho_\infty a_\infty L/\lambda_1$ and $\mathrm{Re}_2 = \rho_\infty a_\infty L/\lambda_2$, and the Peclet number $\mathrm{Pe} = \rho_\infty a_\infty c_p L/k$. Suppose that the reciprocals of these numbers have the same order and are much smaller than unity. In deriving the approximate equations, in all relations we shall retain only the principal terms, neglecting terms having a higher order of smallness. Therefore, in the Navier–Stokes and heat-transfer equations, the viscosity coefficients λ_1, λ_2 and the thermal conductivity k may immediately be taken as constant and equal to their values in the incident flow.

After linearizing the continuity equation, we obtain**

$$\frac{\partial v_n}{\partial n} + \frac{\partial \rho}{\partial n} = 0.$$

From the projection of the Navier–Stokes equation onto the n-axis it follows that

$$\frac{\partial v_n}{\partial n} + \frac{p_\infty}{\rho_\infty a_\infty^2}\frac{\partial p}{\partial n} = 0.$$

Integration of the last two equations leads to the formulas

$$-v_n = \rho = \frac{p_\infty}{\rho_\infty a_\infty^2}\, p, \tag{8}$$

which express the fact that, in the approximation adopted, compression of the gas takes place adiabatically and the Bernoulli integral holds for the entire flow.

Taking relations (8) into account, from the projection of the Navier–Stokes equation onto the τ-axis we derive the condition

$$\frac{\partial v_n}{\partial \tau} = \frac{\partial v_\tau}{\partial n}, \tag{9}$$

* In the limiting case, at infinity the particle velocity may coincide with the critical velocity.

** The primes over dimensionless variables are omitted.

which means the absence of vortices in the flow. Formulas (8) and (9) characterize the motion of ideal media, devoid of viscosity and thermal conductivity.

The influence of dissipative factors must be taken into account when simplifying the heat-transfer equation. We first transform it so as to eliminate first-order small quantities associated with the fluxes of mass and momentum of the substance. Passing in equation (4) from entropy and temperature to density and pressure by means of formulas (5), and combining the resulting expression with the continuity and Navier–Stokes equations in projection on the axes n and τ, we obtain the required relation:

$$\rho\left[(v_n^2 - a^2)\frac{\partial v_n}{\partial n} + v_n v_\tau\left(\frac{\partial v_n}{\partial \tau} + \frac{\partial v_\tau}{\partial n}\right) + (v_\tau^2 - a^2)\frac{\partial v_\tau}{\partial \tau} - a^2\frac{v_\tau - v_n \operatorname{ctg} A_\infty}{\tau - n \operatorname{ctg} A_\infty}\right] =$$

$$= v_n L_n(\lambda_1, \lambda_2) + v_\tau L_\tau(\lambda_1, \lambda_2) - \frac{\alpha a^2}{c_p} L(k, \lambda_1, \lambda_2). \tag{10}$$

Here $L_n(\lambda_1, \lambda_2)$ and $L_\tau(\lambda_1, \lambda_2)$ denote the right-hand sides of the n- and τ-components of the momentum-conservation equation without the terms $\partial p/\partial n$ and $\partial p/\partial \tau$, respectively, and $L(k, \lambda_1, \lambda_2)$ denotes the right-hand side of equation (4).

In the approximation under consideration,

$$da = \left(\frac{\partial a}{\partial \rho_\infty}\right)_s d\rho = \frac{(m_\infty - 1)a_\infty}{\rho_\infty}\, d\rho \qquad \left(m_\infty = -\frac{1}{2\rho_\infty^3 a_\infty^2}\left(\frac{\partial^2 p}{\partial V_\infty^2}\right)_s,\ V = \frac{1}{\rho}\right).$$

Using the last relations, substituting formulas (7) into equation (10), and retaining in it only the principal terms, we obtain

$$2m_\infty \varepsilon v_n \frac{\partial v_n}{\partial n} + \Delta \operatorname{ctg} A_\infty\left(2\frac{\partial v_n}{\partial \tau} + \frac{v_n}{\tau}\right) - \Delta^2\left(\frac{\partial v_\tau}{\partial \tau} + \frac{v_\tau}{\tau}\right) =$$

$$= \frac{1}{\mathrm{Re}}\left(1 + \frac{\varkappa - 1}{\mathrm{Pr}}\right)\frac{\partial^2 v_n}{\partial n^2}. \tag{11}$$

The total Reynolds number Re appearing in equation (11) is connected with the so-called "longitudinal viscosity" by

$$\frac{1}{\mathrm{Re}} = \frac{4}{3}\frac{1}{\mathrm{Re}_1} + \frac{1}{\mathrm{Re}_2},$$

and the Prandtl number Pr is equal to the ratio of the Peclet number Pe to the Reynolds number Re.

Let us consider the principal regimes of gas motion, which differ according to the velocity of the undisturbed flow.

1) Let the flow velocity at infinity v_∞ be greater than the speed of sound a_∞. Then the third term in equation (11) may be neglected in comparison with the second. We have

$$2m_\infty \varepsilon v_n \frac{\partial v_n}{\partial n} + \Delta \operatorname{ctg} A_\infty \left(2\frac{\partial v_n}{\partial \tau} + \frac{v_n}{\tau}\right) = \frac{1}{\mathrm{Re}}\left(1 + \frac{\varkappa - 1}{\mathrm{Pr}}\right)\frac{\partial^2 v_n}{\partial n^2}. \tag{12}$$

Equation (12) contains only one function v_n; therefore it can be solved independently of equation (9), which serves to determine the second unknown function v_τ. Thus the supersonic motion of a gas in the coordinates n, τ is quasi-one-dimensional, which substantially simplifies its mathematical investigation. Moreover, equation (12) exactly coincides with the equation describing the propagation of sound pulses in nonstationary problems with cylindrical symmetry [(2)]. A detailed analysis of the plane analogue of equation (12), without the term $\Delta \operatorname{ctg} A_\infty v_n/\tau$, is contained in the works of Lighthill [(3)] and Hayes [(4)].

To find the asymptotic laws of attenuation of the disturbances introduced by a body of revolution into a supersonic flow of a dissipative gas, it is sufficient-

more precisely, one must set

$$\varepsilon \ll \Delta = \frac{1}{2\,\mathrm{Re}\operatorname{ctg} A_\infty}\left(1 + \frac{\varkappa - 1}{\mathrm{Pr}}\right).$$

Then the first term on the left-hand side of equation (12) may be neglected, and for the function $u = \sqrt{\tau v_n}$, which is proportional to the square root of the energy density transported by the disturbances, one obtains the classical heat-conduction equation. Hence we find [(2)] that the change in the gas parameters in the wave diverging from the body being flowed around obeys the formula

$$v_n = \frac{hn}{\tau^2} e^{-n^2/4\tau} \quad (h = \mathrm{const}). \tag{13}$$

2) If at infinity the flow velocity v_∞ is exactly equal to the speed of sound a_∞, then the second term on the left-hand side of equation (11) vanishes. In this limiting case it is necessary to retain the third term, which does not exert a substantial influence on the structure of purely supersonic flows. For $M_\infty = 1$ the Mach angle $A_\infty = \pi/2$, $\sin A_\infty = 1$, and $\cos A_\infty = 0$. Hence $x = n$, $r = \tau$, $v_x = v_n$, and $v_r = v_\tau$. As a result one obtains the equation

$$2m_\infty \varepsilon v_x \frac{\partial v_x}{\partial x} - \Delta^2 \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} \right) = \frac{1}{\mathrm{Re}} \left(1 + \frac{\varkappa - 1}{\mathrm{Pr}} \right) \frac{\partial^2 v_x}{\partial x^2}, \tag{14}$$

which was studied in [5]. The joint solution of equations (9) and (14) shows that as $r \to \infty$ the maximum excess pressure on streamlines decreases as $r^{-4/3}$, whereas for $M_\infty > 1$ it falls according to formula (13) as $r^{-3/2}$.

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Received
6 IX 1966

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Note: Figure translations are in progress. See original paper for figures.