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Abstract

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SELF-SIMILAR SOLUTIONS OF THE PROBLEM OF RECOMBINATION DECAY OF A PLASMA

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In connection with problems of plasma physics and some others, a number of works have appeared (for example, ([1–3])) devoted to the study of the diffusion equation for an ionized gas in the presence of recombination. In the present work we study self-similar solutions of the equation

$$\frac{\partial n}{\partial t} = D\,\frac{1}{r^{\nu}}\,\frac{\partial}{\partial r}\left(r^{\nu}\frac{\partial n}{\partial r}\right) - \alpha n^{2} \qquad (1)$$

in an infinite region for the case of planar ($\nu = 0$) and spherical ($\nu = 2$) symmetry.* It is assumed that the initial distribution of particles $n_0(r)$ is a bounded function that decreases sufficiently rapidly at infinity.

Equation (1) describes recombination in a plasma when the concentrations of charged particles are sufficiently small (the contribution of triple collisions of these particles to recombination may be neglected); obviously, such concentrations are always reached at sufficiently large times.

Since the initial distributions of particle density, generally speaking, are not self-similar, the solution of the diffusion equation also becomes self-similar as $t \to \infty$. In the linear case such a self-similar solution depends only on the total initial number of particles. For the nonlinear equation (1) the situation turns out to be essentially different.

Let us first consider the case $\nu = 0$. It is easy to verify that equation (1) for $\nu = 0$ admits a self-similar solution of the form

$$n(r,t) = f(\xi)/\alpha(t+\tau); \qquad \xi = r/\sqrt{D(t+\tau)}, \qquad (2)$$

where τ is a constant.

Substituting (2) into equation (1), we obtain an ordinary differential equation for the function $f(\xi)$

$$f'' + \frac{1}{2}\xi f' + f = f^2. \tag{3}$$

Of interest are solutions of (3) that decrease sufficiently rapidly as ξ increases and satisfy the condition

$$f'(0) = 0, \tag{4}$$

since at sufficiently large times the density distribution must become symmetric (even if the initial distribution did not possess this property).

Investigation of equation (3) shows that positive bounded solutions lie in the interval

$$0 \leq f(\xi) \leq 1, \tag{5}$$

* Some remarks on the case $\nu = 1$ are contained in work ([3]).

Moreover, the solution that satisfies condition (4) and has the correct asymptotics corresponds to the unique value $f(0) = 0.690$.

We give the results of numerical integration of equation (3):

ξ	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$f(\xi)$	0.690	0.686	0.673	0.652	0.624	0.588	0.547	0.502

ξ	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
$f(\xi)$	0.453	0.402	0.350	0.300	0.252	0.208	0.168	0.133

ξ	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6
$f(\xi)$	0.103	0.0783	0.0581	0.0422	0.0300	0.0209	0.0142	0.0094

ξ	4.8	5.0	5.2	5.4	5.6	5.8	6.0
$f(\xi)$	0.0061	0.0039	0.0024	0.0015	0.0009	0.0005	0.0003

From the numerical solution presented, one can determine the values of the following integrals:

$$\int_0^\infty f(\xi)\,d\xi = 2\int_0^\infty f^2(\xi)\,d\xi = 1.451; \qquad \int_0^\infty \xi f^2(\xi)\,d\xi = f(0) = 0.690. \quad (6)$$

If the initial distribution does not differ too strongly from the self-similar one, i.e. $N_0 l_0 \sim D/\alpha$ (where l_0 is the characteristic size of the initial distribution), the constant τ entering into (2) can be determined from the condition

$$N_0 = \int_{-\infty}^{\infty} n(r,0)\,dr. \quad (7)$$

Using relations (6), we obtain

$$\tau = 8.421\,D/(\alpha N_0)^2. \quad (8)$$

When the initial distribution is substantially non-self-similar, two cases are possible. For $N_0 l_0 \gg D/\alpha$, the approach to the self-similar regime is determined by the characteristic recombination time $\tau_R \sim l_0^2/\alpha N_0$. In the opposite case $N_0 l_0 \ll D/\alpha$, the determining process in the approach to the self-similar regime will be diffusion, with characteristic time $\tau_D \sim l_0^2/D$.

For $t \gg \tau$, the course of the process ceases to depend on the characteristics of the initial distribution, and the asymptotic law in the problem under consideration proves to be universal for all admissible initial distributions:

$$n(r,t) \underset{t\to\infty}{\simeq} f(r/\sqrt{Dt})/\alpha t, \quad (9)$$

where $f(\xi)$ is the function tabulated above. In the same limiting case $t \gg \tau$, for the decrease of the total number of particles (per unit area) one obtains the expression

$$N(t) \underset{t\to\infty}{\simeq} 2.902\,\sqrt{D}/\alpha\sqrt{t}. \quad (10)$$

The characteristics of the initial distribution N_0 and l_0 determine only the time of approach to the universal regime (9), (10).

As is known, in bimolecular reactions the particle concentration decreases at large times according to the law

$$n(t) = 1/\alpha t. \quad (11)$$

Taking diffusion into account, the change in concentration is described by formula (9). Let us define the mean mass value of the concentration by the relawe find

$$\bar{n}(t) = \int_{-\infty}^{\infty} n^2(r,t)\,dr \Big/ \int_{-\infty}^{\infty} n(r,t)\,dr; \quad (12)$$

then from (6) it follows directly that $\bar{n}(t) = 1/2\alpha t$, i.e., twice smaller than in the case of a spatially uniform distribution of concentrations.

Let us turn to the case of spherical symmetry. In this case the relative role of recombination and diffusion in the change of concentration at the initial stage is likewise determined by the parameter $\varepsilon = \alpha N_0/Dl_0$.

If $\varepsilon \ll 1$, the concentration changes only as a result of diffusion, while the total number of particles remains constant. In this case the particle-density distribution is described by the well-known self-similar formula

$$n(r,t) = \frac{N_0}{(4\pi Dt)^{3/2}}\, e^{-r^2/4Dt}. \quad (13)$$

In the opposite case $\varepsilon \gg 1$, recombination is the determining process until there are reached some total number of particles N_∞ and distribution size l_∞ such that $\alpha N_\infty/Dl_\infty \lesssim 1$. After this, the particle-density distribution has the form (13) with N_∞ instead of N_0. Thus, in contrast to the plane case, in which all particles recombine as $t \to \infty$,* in the spherical case a "quenching" of the ionization occurs. The law of approach to the final state (with total number of particles N_∞) has the form

$$N(t) \underset{t\to\infty}{\simeq} N_\infty \left(1 + \alpha N_\infty \sqrt{2\pi/D^3 t}\right). \quad (14)$$

The residual number of particles N_∞ depends on the initial number of particles N_0, and also on the size and shape of the initial distribution. In the general case there is no simple expression for N_∞.

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REFERENCES

1. G. A. Grinberg, *ZhTF*, **33**, No. 12, 1430 (1963); V. E. Golant, G. A. Grinberg, *ZhTF*, **33**, No. 9, 1139 (1963).

2. E. Gray, D. Kerr, *Ann. Phys.*, **17**, No. 2 (1962).

3. T. L. Perel' man, S. I. Anisimov, *DAN*, **136**, No. 4, 810 (1961).

* According to law (10). In the cylindrical case, correspondingly, $N(t) \sim 1/\ln t$ as $t \to \infty$ (see (3)).

Note: Figure translations are in progress. See original paper for figures.