DUALITY OF FUNCTORS

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Abstract

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DUALITY OF FUNCTORS

IN THE CATEGORY OF HOMOTOPY TYPES

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Duality of functors in the category of topological spaces with a distinguished point was proposed by the author ([1]) as a formalization of Eckmann–Hilton duality. In the present note an analogous duality is studied in the category of weak homotopy types of spaces with distinguished points. Such a change of category makes it possible to eliminate two important shortcomings of the duality operator introduced in ([1]). First, the latter is not homotopical. For example, for any CW-complex there is a homotopy equivalence $X * X \sim \Sigma(X \# X)$ (we borrow the notation from ([1])). But of the functors $X \to X * X$ and $X \to \Sigma(X \# X)$, the first is reflexive, while the second is dual to the zero functor. In the sense of the present note, however, these functors do not differ from one another at all. Secondly, the new approach makes it possible completely to avoid considerable difficulties of a general topological nature that arise in considering the duality operator ([1]).

The plan of the note is as follows. First (§§ 1, 2) the definition is given of the category, functors, and duality; § 3 contains some examples; in § 4 restrictions are formulated under which the equality

$$\mathcal{D}(ST) = \mathcal{D}S \circ \mathcal{D}T$$

holds; § 5 is devoted to consideration of the category of stable homotopy types.

§ 1. An object of the category $\mathcal{H}$ is a class of weakly homotopy equivalent Hausdorff spaces with a distinguished point. In order to specify an object X of the category $\mathcal{H}$, one must specify a space $\bar{X}$ representing it. Two objects are considered equivalent, respectively canonically equivalent, if between the spaces representing them there exists, respectively, a specified weak homotopy equivalence. A morphism $f : X \to Y$ of an object X into an object Y is specified if a continuous map $\bar{f} : \bar{X} \to \bar{Y}$, carrying the distinguished point to the distinguished point, is specified. Two morphisms represented by maps $\bar{f}_i : \bar{X}_i \to \bar{Y}_i$, $i = 1, 2$, are considered canonically equivalent if a weak homotopy equivalence of the pairs $(Z\bar{f}_1, \bar{X}_1)$, $(Z\bar{f}_2, \bar{X}_2)$ is specified. Here Zf is the mapping cylinder of f.

Remark 1. Obviously, the category $\mathcal{H}^{\mathcal{H}}$ of morphisms of the category $\mathcal{H}$ is equivalent to the category $(\mathcal{H}, \mathcal{H})$ of weak-homotopy classes of pairs.

Remark 2. If $*$-morphisms $f : X \to Y$ and $g : Y \to Z$ are specified, then they determine, up to canonical equivalence, a morphism $gf : X \to Z$—their composition.

For any two objects X and Y of the category $\mathcal{H}$ the objects $X \times Y$, $X \vee Y$, $X \# Y$, X^Y are specified (construction of the last: CW-complexes $\bar{X}$ and $\bar{Y}$ representing the objects X and Y are chosen, and the space $\bar{X}^{\bar{Y}}$ is regarded as a representative of X^Y). Obviously, the equality

$$(Y^Z)^X = Y^{Z \# X};$$

holds; specifying a morphism $X \to Y^Z$ is equivalent to specifying a morphism $Z \# X \to Y$.

* "Specified" always means specified up to canonical equivalence.

Definition 1. A covariant functor $S : \mathcal{H} \to \mathcal{H}$ is called **continuous** if, for any two objects $X, Y \in \mathcal{H}$, a morphism $i_S(X, Y) : X \# SY \to S(X \# Y)$ is given, and the correspondence $(X, Y) \mapsto i_S(X, Y)$ defines a covariant functor $i_S : \mathcal{H} \times \mathcal{H} \to \mathcal{H}^H = (\mathcal{H}, \mathcal{H})$.

Definition 2. A natural map $\varphi : S \to T$ of a continuous functor S into a continuous functor T is called **continuous** if the two functors $i_1, i_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{H}^H$, where $i_1(X, Y), i_2(X, Y) : X \# SY \to T(X \# Y)$,

$$i_1(X, Y) = \varphi_{X \# Y} \circ i_S(X, Y), \qquad i_2(X, Y) = i_T(X, Y) \circ [X \# \varphi_Y],$$

are equivalent.

Denote by $\xi : \mathcal{H}^H \to \mathcal{H}$ the functor which assigns to a morphism $f : X \to Y$, represented by a map $\bar{f} : \bar{X} \to \bar{Y}$, the object $Y \cup_f CX$, represented by the space $\bar{Y} \cup_{\bar{f}} C\bar{X}$ (here $C\bar{X}$ is the cone over the space $\bar{X}$).

Remark 3. It is easy to check that if the functor S is continuous, then the functor $S' : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $S' = \xi \circ i_S$, is continuous in each argument. For a continuous map $\varphi : S \to T$, the functor $T \cup_\varphi CS$ ($(T \cup_\varphi CS)X = TX \cup_{\varphi_X} CSX$) is continuous.

Remark 4. The functors and maps of functors defined in (1) define continuous functors and their continuous maps in the sense of the definitions given here. In particular, for any object $A \in \mathcal{H}$ the continuous functors Σ_A and Ω_A are defined, and for any morphism $f : A_1 \to A_2$ the continuous maps $\Sigma_{A_1} \to \Sigma_{A_2}$ and $\Omega_{A_2} \to \Omega_{A_1}$ are defined.

Remark 5. If S, T are continuous functors, then their composition ST is a continuous functor.

Remark 6. In the definition of a continuous functor, the requirement that there exist a map $X\#SY \to S(X\#Y)$ may be replaced by the requirement that there exist a map $X^Y \to SX^{SY}$ with certain natural properties.

§ 2. Let $\mathcal{F}$ be the category whose objects are the continuous functors in $\mathcal{H}$, and whose morphisms are the continuous maps of functors.

Definition 3. A contravariant functor $\mathcal{D} : \mathcal{F} \to \mathcal{F}$ is called a **duality operator** if:

1°. $\mathcal{D}\Sigma_A = \Omega_A$ and $\mathcal{D}\Omega_A = \Sigma_A$ for any $A \in \mathcal{H}$.

2°. There exists a natural map $\varkappa : \mathcal{E} \to \mathcal{D}\mathcal{D}$, where $\mathcal{E}$ is the identity functor in the category $\mathcal{F}$, such that the composition

$$\mathrm{Mor}_F(S, \mathcal{D}T) \to \mathrm{Mor}_F(\mathcal{D}\mathcal{D}T, \mathcal{D}S) \to \mathrm{Mor}_F(T, \mathcal{D}S)$$

is one-to-one and onto for all $S, T \in \mathcal{F}$. Here $\mathrm{Mor}_F(\ ,\ )$ is the set of morphisms from an object to an object in the category $\mathcal{F}$.

Theorem 1. *The duality operator exists and is unique.*

The proof is based on the following lemma.

Lemma. *Let $P : \mathcal{H} \to \mathcal{S}$ be a contravariant functor from the category $\mathcal{H}$ to the category $\mathcal{S}$ of sets with distinguished element $*$, such that:*

1. *$P(\bigvee_{i\in I} X_i) = \prod_{i\in I} P(X_i)$. Here the sign $\bigvee$ denotes a bouquet, $\prod$ a product, and I an arbitrary set of indices.*
2. *If $\alpha \in P(X)$, $f : Y \to X$ is a morphism, and $P(f)\alpha = *$ in $P(Y)$, then there exists $\beta \in P(X \cup_f CY)$ such that $\alpha = P(i)\beta$, where $i : X \to X \cup_f CY$ is the inclusion.*
3. *Let $\beta_1, \beta_2 \in P(X \cup_f CY)$ and $P(i)\beta_1 = P(i)\beta_2$. Then there exists an element $\gamma \in P(\Sigma Y)$ such that*

$$P(\pi)(\beta_1, \gamma) = \beta_2,$$

where

$$\pi : X \cup_f CY \to (X \cup_f CY) \vee \Sigma Y$$

is the separation of the suspension from the cone.

Then there exists an object $A \in \mathcal{H}$ (unique up to canonical equivalence) such that

$$P(X) = \mathrm{Mor}_H(X, A).$$

This lemma follows from the usual obstruction and distinguishing theorems.

Let, further, $S, T \in \mathcal{F}$. Denote, for any $X \in \mathcal{H}$,

$$P(X) = \mathrm{Mor}_F(S, \Omega_{XT}).$$

The functor P satisfies the conditions of the lemma. By $\{S \to T\}$ we shall denote the object of $\mathcal{H}$ existing by this lemma.

It is easily verified that $\{\Sigma_A S \to T\} = \{S \to \Omega_A T\} = \Omega_A\{S \to T\}$; $\{S\Omega_A \to T\} = \{S \to T\Sigma_A\}$; $SA = \{\Omega_A \to S\}$, the equalities meaning canonical equivalence. Continuous mappings $S_1 \to S_2$, $T_1 \to T_2$ induce morphisms $\{S_2 \to T\} \to \{S_1 \to T\}$, $\{S \to T_1\} \to \{S \to T_2\}$. We note that, in checking these assertions, and also later on, it will be necessary to use not only the lemma formulated above, but also the analogous assertion for pairs.

It remains now to put

$$\mathcal{D}SA = \{S \to \Sigma_A\}.$$

If $A_1 \to A_2$ is a morphism, then a continuous mapping $\Sigma_{A_1} \to \Sigma_{A_2}$ is induced and, consequently, a morphism $\mathcal{D}SA_1 \to \mathcal{D}SA_2$. The continuity of the functor $\mathcal{D}S$, as well as the fulfillment of properties 1^0 and 2^0, are checked directly.

In addition to property 2^0 one can prove that there is a natural canonical equivalence, in S and T,

$$\{S \to \mathcal{D}T\} = \{T \to \mathcal{D}S\}.$$

§ 3. **Hypothesis.** The functor S is reflexive (i.e. $\mathcal{D}^2 S = S$) if and only if there exists a reflexive functor $\overline{S}$ in the category of topological spaces with distinguished point such that the functor is obtained from the functor S by passing from topological spaces to their weak homotopy types (see Remark 4). In this case the functor is obtained in the same way from DS.*

From this hypothesis it would follow, for the operator D from (1), that if $\overline{S}_1$ and $\overline{S}_2$ are two reflexive functors in the category of topological spaces and if for every CW-complex A the spaces $\overline{S}_1 A$ and $\overline{S}_2 A$ are weakly homotopy equivalent naturally in A, then the same is true also for the spaces $D\overline{S}_1 A$ and $D\overline{S}_2 A$. For nonreflexive functors this is false, as is shown by the example given in the introduction.

The stated hypothesis does not contradict the existence of nonreflexive functors in the category of topological spaces which become reflexive upon passage to the category $\mathcal{H}$. Here are two examples: P, where $P(X) = X \# X$, and T_3, where $T_3 X \subset X \times X \times X$; $T_3(X) = \{(x_1, x_2, x_3), x_1 = *, \text{ or } x_2 = *, \text{ or } x_3 = *\}$. The reflexivity of these functors in the category $\mathcal{H}$ would follow immediately from the stated hypothesis, but it can also be proved without it by a direct computation of the dual functor. Namely, the functor $\mathcal{D}P$ assigns to the weak-homotopy class of a space A the weak-homotopy class of the space of paths in $A \times A$ beginning at the distinguished point and ending at points of the bouquet $A \vee A \subset A \times A$. The space $\mathcal{D}T_3 A$ can be obtained by contracting, in the direct product $A \times \mathcal{D}PA$, the subspace $* \times \mathcal{D}PA$ to a point.

One can also check that the functor J, $JX = X * X$, is reflexive and that, with respect to the operator $\mathcal{D}$, the space $\mathcal{D}JA$ has the homotopy type $DJA = K(A, A)$. Hence it follows that

$$K(A, A) \sim \Omega(\mathcal{D}PA).$$

§ 4. Consider continuous functors satisfying the following additional condition:

(∗) Let $A, C, X \in \mathcal{H}$ be arbitrary objects. Suppose that for all $B \in \mathcal{H}$ a morphism

$$\rho_B : X \to A\#(B\#C)^{SB}$$

is given such that, if $\varphi : B_1 \to B_2$ is a morphism, then

$$(A\#(\varphi\#C)^{SB_1}) \circ \rho_{B_1} = (A\#(B_2\#C)^{S\varphi}) \circ \rho_{B_2}.$$

Then there exists a morphism

$$\rho : X \to A\#\{S \to \Sigma_C\}$$

such that for all $B \in \mathcal{H}$ there is a canonical equivalence

$$(A\#\pi_B) \circ \rho = \rho_B,$$

where

$$\pi_B : \{S \to \Sigma_C\} \to (C\#B)^{SB}$$

is the obvious morphism.

Theorem 2. If $S, T \in \mathcal{F}$, and the functor T satisfies axiom (∗), then

$$\mathcal{D}(S \circ T) = \mathcal{D}S \circ \mathcal{D}T.$$

Proof essentially repeats verbatim the proof of the analogous assertion in (2). It differs from it in two places:

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* By the symbol D, in contrast to $\mathcal{D}$, we denote the duality operator from note (1).

first, all the difficulties arising from point-set topology are removed. Secondly, in constructing the mapping $\mathcal{D}(\Omega_A T) \to \Sigma_A \mathcal{D}T$, at the very end the fulfillment of axiom $(*)$ for T is required.

Condition $(*)$ is fulfilled, for example, for $S = \Sigma_A$, since in this case $(B\#C)^{SB} = \{S \to \Sigma_C\}$ for $B = \Gamma$ (a two-point set). Similar considerations make it possible to verify axiom $(*)$ for all the basic functors from note (1). But whether it is always fulfilled, I do not know.

§ 5. Duality can be defined in exactly the same way also in the category of stable homotopy types. For them it turns out:

Theorem 3. *In the category $\mathcal{H}^{(S)}$ of stable weak homotopy types, every continuous functor is a functor of the form Σ_A. Moreover,*

$$\mathcal{D}(\Sigma_A) = \Sigma_B,$$

where B is dual to A in the Spanier-Whitehead sense.

The proof is quite simple.

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Note: Figure translations are in progress. See original paper for figures.