

The number of periodic solutions of the second kind to the differential equation $\ddot{\varphi} - f_1(\varphi)\dot{\varphi} - f_0(\varphi) = 0$

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Abstract

The equation

$$\ddot{\varphi} - f_1(\varphi)\dot{\varphi} - f_0(\varphi) = 0 \quad (1)$$

is considered in the case where $f_0(\varphi)$ and $f_1(\varphi)$ are trigonometric polynomials of degree no higher than n , where n is a fixed natural number. Equation (1) is equivalent to the system

$$\frac{d\varphi}{dt} = z, \quad \frac{dz}{dt} = f_0(\varphi) + f_1(\varphi)z, \quad (2)$$

, for which the phase space is a circular cylinder. A periodic solution of the second kind of equation (1) corresponds to a closed trajectory of system (2) encircling the phase cylinder, and vice versa. It is proved that for any n , there exist polynomials $f_0(\varphi)$ and $f_1(\varphi)$ for which the number of such trajectories is at least n . Bibliography: 12 items.

Full Text

Preamble

This section investigates the existence and stability of periodic solutions for a class of differential equations. We consider the equation:

$$\frac{d\phi}{dt} = 1, \quad \frac{dz}{dt} = f_1(\phi)z + f_0(\phi)z^2 + \dots$$

where $f_i(\phi)$ are periodic functions with period 2π . Following the methodologies established in [?, ?], we analyze the behavior of the system near $z = 0$. The functions $f_1(\phi)$ and $f_0(\phi)$ are assumed to be sufficiently smooth, and we specifically examine cases where $f_1(\phi) = \cos \phi$ or involves higher-order trigonometric terms such as $\sin(2k + 1)\phi$ and $\cos 2k\phi$.

1. Transformation and Expansion

To analyze the trajectories, we introduce the transformation $z = \rho u(\phi, \rho)$, where $u(\phi, 0) = 1$. Substituting this into the governing equations, we derive a power series expansion for the solution in terms of the initial displacement ρ_0 . The displacement function after a full period 2π is given by:

$$F(\rho, 2\pi, a) - F(\rho, 0, a) = \sum_{k=2}^{\infty} \alpha_k(2\pi, a) \rho^k$$

where the coefficients α_k depend on the parameters $a = (a_1, a_2, \dots, a_n)$. The vanishing of these coefficients determines the existence of periodic solutions (limit cycles) in the vicinity of the equilibrium point.

We define the auxiliary functions $u_k(\phi)$ and $v_{kl}(\phi)$ to satisfy the recursive system of linear differential equations:

$$\begin{aligned} u'_1(\phi) &= 0, & u_1(0) &= 1 \\ u'_l(\phi) &= (l-1)u_{l-1}(\phi)f_1(\phi), & (l &= 2, 3, \dots) \end{aligned}$$

For the specific case where $f_1(\phi) = \cos \phi$, the solutions take the form $u_k(\phi) = \sin^{k-1} \phi$. This allows for the explicit calculation of the focal values and the determination of the stability of the origin.

2. Analysis of Focal Values

The coefficients $\alpha_k(2\pi, a)$ are calculated using the properties of the functions $v_{kl}(\phi)$. For $m = 1, 2, \dots$, we consider the case where $f_1(\phi) = \cos \phi$ and $f_0(\phi)$ is a trigonometric polynomial. The integration over the period 2π yields:

$$\alpha_{2m+2}(2\pi, a) = \frac{(-1)^m \pi}{2^{2m-2}} a_{2m+1} + \dots$$

The coefficients α_k for $k < 2m + 2$ vanish under specific parameter constraints, indicating a higher-order focus. By varying the parameters a_k , we can bifurcate multiple limit cycles from the origin. The relationship between the coefficients a_k and the resulting displacement function is analyzed to ensure the independence of the bifurcating cycles.

3. Special Cases and Trigonometric Polynomials

We further examine the scenario where $f_1(\phi) = \cos \phi$ and $f_{0k}(\phi) = \cos(2m-2)\phi$. Using the recursive relations (12) and (14), we determine the conditions under which $\alpha_k(2\pi, a) = 0$. For $m \geq 2$, the functions $v_{kl}(\phi)$ are expressed as:

$$v_{31}(\phi) = a_{30} \cos(2m-1)\phi + \dots$$

The resulting algebraic system for the coefficients b_{k0} allows us to identify the maximum number of periodic solutions. Specifically, we demonstrate that for a given degree of the trigonometric polynomial $f_0(\phi)$, there exists a neighborhood of the origin in the parameter space where the system exhibits exactly n limit cycles.

4. Conclusion and Stability

The stability of the periodic solutions is determined by the sign of the first non-vanishing coefficient $\alpha_k(2\pi, a)$. If $\alpha_k < 0$, the limit cycle is stable; if $\alpha_k > 0$, it is unstable. The results obtained here extend the classical theorems of Lyapunov and Poincaré regarding the center-focus problem to non-autonomous systems with periodic coefficients. The construction of the displacement function $F(\rho, \phi, a)$ provides a robust framework for studying bifurcations in higher-dimensional parameter spaces.

References

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Note: Figure translations are in progress. See original paper for figures.

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