Systems of differential equations with algebraic moving singular points

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Abstract

A system of differential equations

$$\frac{dx}{dz} = \sum_{j=0}^{p} a_j(z) y^{p-j}, \quad \frac{dy}{dz} = \sum_{j=0}^{k} b_j(z) x^{k-j}, \tag{1}$$

is considered, where a_j, b_j are holomorphic functions and $k \geq p \geq 2$. Necessary and sufficient conditions are provided for the movable singular points of the system to be algebraic, and the nature of movable singular points is investigated if they are not algebraic. It is established that systems with constant coefficients and with variables for which the condition $k+1 \neq M(p+1)$ (M is an integer) is satisfied have only algebraic movable singular points. However, for k and p other than $k=p=2$; $k=p=3$; $k=5$, $p=2$, for which $k+1=M(p+1)$, the condition for the algebraicity of movable singularities generally includes the values of $b_j(z)$ and their first derivatives ($j = 0, 1, 2, \ldots, M+1$). For the cases $k=p=2$; $k=p=3$; $k=5$, $p=2$, coefficients that ensure the algebraicity of all movable singularities are found. Bibliography: 8 items.

Full Text

Introduction and Problem Statement

This work investigates the asymptotic behavior of solutions to systems of differential equations near singular points. We consider a system of the form:

$$\begin{aligned} \frac{dx}{dz} &= a(z, x, y) \\ \frac{dy}{dz} &= b(z, x, y) \end{aligned} \tag{1}$$

where the functions $a(z, x, y)$ and $b(z, x, y)$ are defined in a domain D and possess specific algebraic structures. Specifically, we examine cases where $(k-1)(p-1) > 1$ for integers k and p. Near a point $z_0 \in D$, we assume the functions can be represented as expansions:

$$a(z, x, y) = \sum a_{ij}(z) x^i y^j, \quad b(z, x, y) = \sum b_{ij}(z) x^i y^j$$

with leading coefficients a_{k0} and b_{0p} being non-zero. Our objective is to determine the conditions under which there exist solutions $x(z)$ and $y(z)$ that vanish as $z \to z_0$.

Asymptotic Analysis of Solutions

By substituting formal power series into the system (1), we seek solutions of the form:

$$x(z) = x_0 (z - z_0)^s, \quad y(z) = y_0 (z - z_0)^t \tag{3}$$

where the exponents s and t are determined by the leading terms of the equations. Balancing the orders of magnitude leads to the relation $ps+r = tp$, where r and t are constants related to the indices k and p. For the existence of non-trivial coefficients x_0 and y_0, the following algebraic relations must be satisfied:

$$x_0 = b_{00}[(k+1)p(p+1)]^{-1}$$
$$y_0 = a_{00}[(p+1)k(k+1)]^{-1}$$

These coefficients are valid provided the denominators do not vanish and specific constraints on the parameters k and p are met. In particular, we find that for $n > 0$, the solutions remain stable in the neighborhood of z_0.

Transformation to Canonical Form

To further analyze the stability and convergence of these asymptotic expansions, we introduce a change of variables:

$$u = x(z)(z-z_0)^{-s}, \quad v = y(z)(z-z_0)^{-t} \tag{12}$$

This transformation yields a new system for u and v:

$$(z-z_0)\frac{du}{dz} = -su + F_1(u,v,z)$$
$$(z-z_0)\frac{dv}{dz} = -tv + F_2(u,v,z)$$

where F_1 and F_2 are higher-order terms that vanish as $(u,v) \to (0,0)$. As $z \to z_0$, the behavior of the system is governed by the linear part, and the existence of a holomorphic or asymptotic solution depends on the eigenvalues of the associated Jacobian matrix. Following the methods established by Horn [**?**] and Briot-Bouquet, we can demonstrate that if the real parts of the exponents are positive, there exists a unique trajectory $L_0(z)$ approaching the origin.

Conditions for Existence and Uniqueness

The existence of the limit $(x(z), y(z)) \to (0,0)$ as $z \to z_0$ along a path $L_0(z)$ requires that the integral of the leading terms converges. Specifically, we require:

$$\int_{L_0} x^k y^p dz < \infty$$

Under these conditions, the asymptotic representation of the solution near the singularity z_0 can be expressed as:

$$\begin{aligned} x(z) &= x_0(z-z_0)^s(1+\gamma_1(z)) \\ y(z) &= y_0(z-z_0)^t(1+\gamma_2(z)) \end{aligned} \tag{23}$$

where $\gamma_1, \gamma_2 \to 0$ as $z \to z_0$. In cases where the parameters k and p satisfy specific integer relations, logarithmic terms of the form $\ln(z-z_0)$ may appear in the expansion.

Analysis of Parameter Constraints

We examine several specific cases for the indices k and p. For instance, if $k = p = 3$, the system exhibits a high degree of symmetry. If $k = 5$ and $p = 2$, the constraints on the coefficients a_{ij} and b_{ij} become more restrictive. We define a characteristic polynomial Q based on the coefficients a_{k0} and b_{0p}. The non-vanishing of Q is a necessary condition for the existence of the described asymptotic branches.

For the case $k = p = 2$, which has been studied extensively in previous literature [**?**, **?**], the solutions are generally well-behaved. However, for higher-order singularities where $(k+1)(p-1) + (p+1)(\gamma - 1) = 0$, the structure of the solution space becomes significantly more complex, potentially involving multiple branching points in the complex plane.

Conclusion

The analysis demonstrates that the singular point z_0 acts as a focal point for a family of solutions whose behavior is determined by the leading algebraic terms of the system. The results extend the classical theory of Briot-Bouquet to more general non-linear systems where the leading powers are not necessarily unity. Further investigation is required to characterize the global behavior of these solutions as they move away from the neighborhood of z_0 into the broader domain D.

References

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Note: Figure translations are in progress. See original paper for figures.