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Abstract

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MATHEMATICS

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COMPLETENESS OF TRANSLATES OF A FUNCTION IN CERTAIN SPACES

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1°. The completeness* of the system of functions $\{f_x(\tau)\}$, $f_x(\tau) = f(x+\tau)$, $-\infty < x < \infty$, in the spaces L_α,

$$\|f\| = \int_{-\infty}^{\infty} |f(\tau)| e^{\alpha(\tau)}\, d\tau,$$

where $\alpha(\tau) = \alpha(-\tau) > 0$, $\alpha(\tau)$ is nondecreasing, $\alpha(\tau)/\tau$ is nonincreasing, $\alpha(\tau_1 + \tau_2) \le \alpha(\tau_1) + \alpha(\tau_2)$, was studied in ($^{1-3}$). It turned out that the conditions for completeness depend on the rate of growth of $\alpha(\tau)$. Shilov (^{2}) showed that, for $\alpha(\tau) = o(\ln\tau)$, the system $\{f_x(\tau)\}$ is complete in L_α if and only if

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{ix\tau}\, d\tau \ne 0, \qquad -\infty < x < \infty. \tag{1}$$

Korenblum (^{3}) studied the case $\alpha(\tau) = c\tau$. He found that, instead of (1), the necessary and sufficient conditions** are

$$\tilde{f}(z) \ne 0, \qquad |\operatorname{Im} z| \le c, \qquad -\infty < \operatorname{Re} z < \infty,$$

$$\varlimsup_{x\to\infty} \ln|\tilde{f}(x)|/e^{\pi x/2c} = 0, \qquad \varlimsup_{x\to-\infty} \ln|\tilde{f}(x)|/e^{-\pi x/2c} = 0. \tag{2}$$

The question naturally arises: for which $\alpha(\tau)$ is condition (1) sufficient for the completeness of the system $\{f_x(\tau)\}$ in L_α. Let N be the class of such functions $\alpha(\tau)$. In the present paper a sufficient condition is obtained for $\alpha(\tau)$ to belong to the class N, which is necessary if $\alpha(\tau) = o(\tau)$ and $\tau\alpha'(\tau)/\alpha(\tau)$ is nonincreasing. It is shown that this condition is at the same time, in an analogous way, necessary and sufficient for the regularity of L_α, regarded as a normed ring with multiplication given by convolution. For nonregular rings L, necessary completeness conditions are obtained which are analogous to conditions (2).

2°. Theorem 1. *If*

$$\int_0^\infty \frac{\alpha(\tau)\,d\tau}{1+\tau^2} < \infty, \tag{3}$$

then the ring L_α is regular and the system $\{f_x(\tau)\}$ is complete in L_α if and only if (1) holds.

Proof. We first show the regularity of L_α. From the description of the maximal ideals of L_α obtained in (^{4}) it follows that it suffices, for arbitrary x_0 and $\varepsilon > 0$, to construct a function $\varphi(x) \in L_\alpha$ such that $\tilde{\varphi}(x_0) \neq 0$ and $\tilde{\varphi}(x) = 0$ for $|x - x_0| \geq \varepsilon$. We may assume that $x_0 = 0$ and

$$\alpha(x) \leq \int_1^x \frac{\alpha(\tau)\,d\tau}{\tau}$$

* A system $\{x_\nu\}$ of elements of a Banach space X is called **complete** in X if the closure of the linear span of the system coincides with X.

** In what follows, $\tilde{f}(x)$ denotes the Fourier transform of the function $f(x)$, defined by the equality in formula (1).

for $x \geqslant 2$. Put

$$\beta(\tau) = \alpha(\tau)\left(\int_\tau^\infty \frac{\alpha(\theta)\,d\theta}{1+\theta^2}\right)^{-1/2}, \qquad \tau \geqslant 0, \qquad \beta(-\tau) = \beta(\tau).$$

Using the condition imposed on $\alpha(\tau)$, we obtain, for $A > 1$,

$$\int_0^\infty \frac{\beta(\tau)\,d\tau}{1+\tau^2} < \infty; \qquad \beta(A\tau) \leqslant A^2\beta(\tau); \qquad \lim_{x\to\infty}\left(\int_1^x \frac{\alpha(\tau)\,d\tau}{\tau} - \int_1^{x/A} \frac{\beta(\tau)\,d\tau}{\tau}\right) = -\infty. \tag{4}$$

Next, let $\omega(\tau)$ be the function inverse to $\beta(\tau)$, and for $x \geqslant 1$

$$K(x) = \exp\left(\int_1^x \frac{\alpha(\tau)\,d\tau}{\tau}\right), \qquad N(x) = \exp\left(\int_0^{\omega(x)} \beta'(\tau)\ln\tau\,d\tau\right), \qquad F(x) = \min_{\tau\geqslant 1} N(\tau)x^{-\tau}.$$

Then $N'(\beta(x))/N(\beta(x)) = \ln x$, and therefore $F(x) = N(\beta(x))x^{-\beta(x)}$, i.e.

$$F(x) = \exp\left(-\int_1^x \frac{\beta(\tau)\,d\tau}{\tau}\right). \tag{5}$$

Moreover, for $x \geqslant 1$ we have

$$\frac{N'(x)}{N(x)} = \ln\omega(x), \qquad \ln\frac{N(x+1)}{N(x)} \geqslant \ln\omega(x), \qquad \frac{N(x)}{N(x+1)} \leqslant \int_{x-1}^x \frac{d\theta}{\omega(\theta)},$$

hence, by virtue of (4),

$$\sum_{n=2}^{\infty} \frac{N(n)}{N(n+1)} \leqslant \int_0^\infty \frac{d\theta}{\omega(\theta)} = O(1) + \int_0^\infty \frac{\beta(\tau)\,d\tau}{1+\tau^2} < \infty. \tag{6}$$

Now put

$$\mu_2 = \mu_3 = 1/N(2), \qquad \mu_j = N(j-3)/N(j-2),$$

$$4 \leqslant j < \infty \quad \text{and} \quad \psi(x) = \prod_{n=2}^{\infty} \frac{\sin\mu_n x}{\mu_n x}, \qquad \psi_\delta(x) = \psi(\delta x).$$

By (6), $\psi(x)$ is an entire function of some finite degree. Since

$$|\psi(x)| = O(1)\min_{j\geqslant 2} N(j-1)|x|^{-j},$$

we have $|\psi(x)| = O(F(|x|))$. Hence, from (4), (5) it follows that

$$\int_{-\infty}^{\infty} |\psi_\delta(x)| e^{\alpha(x)}\,dx = O(1)\int_1^\infty F(\delta x)\exp\left(\int_1^x \frac{\alpha(\tau)\,d\tau}{\tau}\right)dx < \infty.$$

Thus $\psi_\delta(x) \in L_\alpha$ for $\delta > 0$, and from the Wiener-Paley theorem we obtain that $\widetilde{\psi}_\delta(x) = 0$ for $|x| \geqslant \delta\sigma$, while $\widetilde{\psi}_\delta(0) = 0$. Therefore the function $\varphi(x) = \psi_{\varepsilon/\sigma}(x)$ is the desired one, and the regularity of L_α is proved.

From the regularity of L_α and the results of G. E. Shilov (2) it follows that, in order to prove the second assertion, it is enough to show the density in L_α of functions with finite Fourier transforms. Let $p(x) \in L_\alpha$. Put

$$p_A(x) = Aq\int_{-\infty}^{\infty} p(x-t)\psi_A(t)\,dt, \qquad q = \left(\int_{-\infty}^{\infty}\psi(t)\,dt\right)^{-1}.$$

Obviously, $p_A(x) \in L_\alpha$ and $\tilde{p}_A(x)$ is finite. We shall show that

$$\lim_{A\to\infty} \|p - p_A\| = 0.$$

For $\mu > 0$ and $A \to \infty$ we have

$$\int_{-\infty}^{\infty} \int_{|t|>\mu} |p(x) - p(x-t)|\, |\psi_A(t)|\, Aq\, dt\, e^{\alpha(x)}\, dx = O(1)\|p\| \int_{|t|>A\mu} |\psi(t)| e^{\alpha(t)}\, dt,$$

$$\int_{-\infty}^{\infty} \int_{|t|<\mu} |p(x) - p(x-t)|\, |\psi_A(t)|\, Aq\, dt\, e^{\alpha(x)}\, dx \leqslant$$

$$\leqslant \int_{|t|<\mu} |\psi_A(t)|\, Aq \int_{-\infty}^{\infty} \left|p(x)e^{\alpha(x)} - p(x-t)e^{\alpha(x-t)}\right| dx\, dt +$$

$$+ \max_{|x|<\infty,\ |t|<\mu} \left|e^{\alpha(x)-\alpha(x-t)} - 1\right| \int_{|t|<\mu} |\psi_A(t)|\, Aq\, dt = o(1) + O(\mu).$$

Consequently,

$$\overline{\lim}_{A\to\infty} \|p - p_A\| = O(\mu).$$

Letting $\mu \to 0$, we obtain

$$\lim_{A\to\infty} \|p - p_A\| = 0,$$

as was required.

3°. **Theorem 2.** *Let $\alpha(\tau) = o(\tau)$, let $\tau\alpha'(\tau)/\alpha(\tau)$ be nondecreasing,*

$$\int_0^{\infty} \frac{\alpha(\tau)\, d\tau}{1+\tau^2} = \infty.$$

Then the Fourier transforms of functions from L_α form a quasi-analytic class of functions, and the ring L_α is nonregular.

Theorem 3. *If $\alpha(\tau)$ satisfies the conditions of Theorem 2 and the system $\{f_x(\tau)\}$ is complete in L_α, then (1) holds and, for real A,*

$$\overline{\lim}_{x\to\infty} \frac{\ln|\tilde{f}(x)|}{\xi(\frac{1}{2}\pi(x-A))} = 0, \qquad \overline{\lim}_{x\to-\infty} \frac{\ln|\tilde{f}(x)|}{\xi(\frac{1}{2}\pi(x+A))} = 0, \tag{7}$$

where $\xi(x)$ is the inverse function of the function

$$\eta(x) = \int_0^x \frac{\alpha(\tau)\,d\tau}{1+\tau^2}.$$

Let us note that (7) coincides with (2) when $\alpha(x) = cx$. Denote by $I_{\gamma,A}$ the set of functions $f \in L_\alpha$ for which

$$\overline{\lim}_{x\to\infty} \frac{\ln|\hat{f}(x)|}{\xi(\frac{1}{2}\pi(x-A))} \leqslant \gamma < 0.$$

Theorem 3 follows from the following assertion, which is of independent interest.

Theorem 4. *Let $\alpha(\tau)$ satisfy the conditions of Theorem 2. Then for arbitrary γ and A the set $I_{\gamma,A}$ contains a function whose Fourier transform does not vanish, and there exists a function $g(t)$, for which*

$$0 < \operatorname{vrai\,sup}_{-\infty<\tau<\infty} \left|g(\tau)e^{-\alpha(\tau)}\right| < \infty, \qquad \int_{-\infty}^{\infty} g(x-t)f(t)\,dt \equiv 0 \tag{8}$$

for all $f \in I_{\gamma,A}$.

We shall need the following lemmas.

Lemma 1. *Let $\omega(t)$ be the inverse function for the function $2\alpha(t)/\pi$,*

$$z = x + iy, \qquad \Phi(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{\omega(n)}\right)e^{-z/\omega(n)}.$$

Then we have

$$\frac{c_1}{1+|y|^{c_2}} \leqslant \Phi(iy)e^{-\alpha(y)} \leqslant \Phi_1(y), \qquad \Phi_1(y) = \Phi_1(-y);$$

$$\Phi_1(y)\uparrow, \qquad \int_0^\infty \frac{\ln\Phi_1(y)}{1+y^2}\,dy < \infty, \tag{9}$$

and, for $x \geqslant 0$,

$$|\Phi(x)| \leqslant c_4 e^{c_3 x}\exp\left(-\frac{2x}{\pi}\int_0^x \frac{\alpha(\tau)\,d\tau}{1+\tau^2}\right), \tag{10}$$

$$|\Phi(z)| \geqslant \frac{c_5}{1+|y|^{c_6}}e^{-c_7 x}\exp\left(-\frac{2x}{\pi}\int_0^x \frac{\alpha(\tau)\,d\tau}{1+\tau^2}\right), \tag{11}$$

where c_i are positive constants.

* $1/g(t)$ depends on γ and A.

Lemma 2. If $f \in I_{\gamma,A}$, then for $x \geqslant 0$

$$\int_0^\infty |\tilde{f}(p)| e^{xp}\, dp \leqslant c_9 \exp\left(\frac{2x}{\pi}\int_0^x \frac{\alpha(\tau)\,d\tau}{1+\tau^2} + c_8 x\right),$$

where c_8 depends only on γ, A.

Proof of Theorem 3. Put

$$R(y) = \left(\frac{\sin y}{y}\right)^B \frac{1}{\Phi(iy)}.$$

By virtue of 9, for sufficiently large B, $R(y) \in L_\alpha$, and from (11) it follows that, for $x > 0,\ \sigma > 0$,

$$|\widetilde{R}(x)| \leqslant \exp\left(-\sigma x + c_{10}\sigma + \frac{2\sigma}{\pi}\int_0^\sigma \frac{\alpha(\tau)\,d\tau}{1+\tau^2}\right).$$

Taking $\sigma = \xi(1/2\pi(x-H))$, for large H we obtain

$$|\overline{R}(x)| \leqslant \exp\left(-\frac{1}{2}H\,\xi(1/2\pi(x-H))\right).$$

It is not difficult to show that, for any $\delta > 0$,

$$\lim_{x=\infty} \frac{\xi(x-\delta)}{\xi(x)} = 0.$$

Therefore one can choose η so that

$$\widetilde{H}(y) = e^{i\eta y} R(y) \in I_{\gamma,A},$$

and $\widetilde{H}(y)$ does not vanish. The first assertion of the theorem is proved.

Next, put

$$\mu(z) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\ln \Phi_1(\rho)\,d\rho}{i\rho - (z+1)}.$$

Then $\mu(z)$ is holomorphic for $\operatorname{Re} z > -1$, and by virtue of (9)

$$\operatorname{Re}\mu(z) \leqslant -\frac{z\ln\Phi_1(z)}{\pi}\int_{\rho>|z|}\frac{(x+1)\,d\rho}{(\rho-y)^2+(x+1)^2}.$$

The function $s(z)=e^{\mu(z)}$ is holomorphic and bounded for $\operatorname{Re} z \geqslant 0$, and

$$|s(iy)| \leqslant \frac{1}{\Phi_1(|y|)}.$$

Put

$$f \in I_{\gamma,A} \quad \text{and} \quad f_1(x)=\int_{-\infty}^{\infty} e^{-(x-\tau)^2} f(\tau)\,d\tau.$$

In this case

$$\max_{-\infty<x<\infty} |f_1(x)e^{\alpha(x)}| < \infty, \qquad f_1(x) \in I_{\gamma,A},$$

and, by Lemma 2, for $x \geqslant 0$,

$$|f_1(iz)| = O(1)\left|\int_{-\infty}^{\infty} \tilde{f}_1(p)e^{zp}\,dp\right| = O(1)\exp\left(\frac{2x}{\pi}\int_0^x \frac{\alpha(\tau)\,d\tau}{1+\tau^2}+c_8x\right).$$

Hence, and from (9), it follows that for any real β the function

$$Q_\beta(z)=\Phi(z)s(z)e^{-2c_8z}f_1(iz+\beta)$$

is holomorphic in the right half-plane, bounded on the axis y and on the positive semiaxis x, continuous for $x \geqslant 0$, and of order $\rho < 2$. Applying the Phragmén-Lindelöf principle, we find that $Q_\beta(z)$ is bounded in the right half-plane. Then, by Cauchy’s theorem,

$$\int_{-\infty}^{\infty}\frac{Q_\beta(iy)\,dy}{(1+iy)^2}=0.$$

Consequently, for $-\infty < \beta < \infty$,

$$\int_{-\infty}^{\infty} \Phi(iy)s(iy)e^{-2c_8iy}f_1(\beta-y)\frac{dy}{(1-iy)^2} = \int_{-\infty}^{\infty} e^{-(\beta-\tau)^2}\int_{-\infty}^{\infty} f(\tau-v)g(v)\,dv\,d\tau = 0, \tag{12}$$

where

$$g(v) = \Phi(iv)s(iv)e^{-2ic_8 v}(1+iv)^{-2}.$$

From (12), by the usual device, we derive

$$\int_{-\infty}^{\infty} f(\tau - v)g(v)\,dv = 0.$$

Since g does not depend on f and, by virtue of (10),

$$0 < \max_{-\infty<v<\infty} |g(v)e^{-\alpha(v)}| < \infty,$$

the last equality is equivalent to the last assertion of Theorem 3.

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Note: Figure translations are in progress. See original paper for figures.

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